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Dimensionally regularized one-loop tensor integrals with massless internal particles

Y. Kurihara[a]

High Energy Accelerator Research Organization, Oho 1-1, Tsukuba, Ibaraki 305-0801, Japan

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Abstract. A set of one-loop vertex and box tensor integrals with massless internal particles has been obtained directly without any reduction method to scalar integrals. The results with one or two massive external lines for the vertex integral and zero or one massive external lines for the box integral are shown in this report. Dimensional regularization is employed to treat any soft and collinear (IR) divergence. A series expansion of tensor integrals with respect to an extra space-time dimension for the dimensional regularization is also given. The results are expressed by very short formulas in a manner suitable for a numerical calculation.

1 Introduction

The LHC (Large Hadron Collider) project [1] at CERN is planned as the next-generation energy-frontier experiment. One of the physics motivations of LHC experiments is to discover the Higgs particle, which is the only one missing ingredient in the standard model. In order to establish the model completely, it is essential to find it and to investigate it in detail. It should also be important to search for new phenomena beyond the standard model through any tiny deviation in experimental observations from theoretical predictions. Further, not only searching for new phenomena, but also performing precise measurements of parameters included in the standard model is another important issue of LHC.

LHC has employed colliding proton–proton beams in order to achieve beam energies as high as possible, which should enhance the possibility to find new particles/phenomena. However, a proton machine must have a large QCD background, since the proton is a composite particle constructed by strongly interacting particles, such as quarks and gluons. This veils signals with large backgrounds. In order to extract as much physics information as possible from experimental data contaminated by huge backgrounds, the behavior of the background should be understood in detail. This implies that one should precisely understand QCD, because it entirely governs the background.

The background of any proton–proton colliding experiment must be completely predicted by QCD, in principle. However, in fact, it is a very difficult task to make precise predictions because of the large coupling constant of QCD. Moreover the lowest level (tree-level) calculation does not have any predictive power for the event rate, because there is no good renormalization point well-defined experimentally. We need higher order perturbation calculations for precise predictions of the background behavior.

Loop integration is one of the critical issues for computing these higher order corrections. In general, N-point tensor and scalar integrals with massless internal lines including an infrared (IR) divergence must be calculated in QCD. Since an arbitrary number of dimensions must be used in QCD to regulate any IR divergence, the usual method for the standard model [2] cannot be used directly. Various methods to reduce $(N \geq 5)$-point integrals into $(N-1)$-point integrals [4] with a dimensionally regulated scheme are proposed. Then, all of the necessary $(N \geq 5)$-point integrals can be expressed by a linear combination of box (4-point) tensor integrals. Usually, box tensor integrals are further reduced to 4- or less point scalar integrals, and then numerically evaluated to obtain higher order corrections. The IR finite box integrals are obtained in [5]; for the IR divergent case, box integrals with zero or one external mass are given in [6]. All IR divergent box integrals are treated in [7] using the partial differential equation method. Another approach to all possible box scalar integrals with massless internal lines with zero to four external massive lines is proposed for the IR divergent case [8] and the IR finite case [9]. Recently, two independent formalisms were proposed for calculating one-loop virtual corrections with an arbitrary number of external legs [10,11].

We propose a new method to calculate tensor integrals directly, and not to use a reduction method from tensor integrals to scalar ones in this report. A set of one-loop vertex and box tensor integrals with massless internal particles is given in terms of hypergeometric functions. Dimensional regularization is employed to treat any IR divergence. A series expansion of tensor integrals with respect to an extra

[a] e-mail: yoshimasa.kurihara@kek.jp

space-time dimension in the dimensional regularization is also given in this report. All results are expressed by very short formulas with a suitable manner for numerical calculations.

A general form of 3-point tensor integrals is given in Sect. 2, and that of 4-point ones in Sect. 3. The obtained results are numerically checked in Sect. 4. A numerical calculation method for the hypergeometric function and their series expansion with respect to an extra space-time dimension is given in Appendix A. The series expansion of the general form of tensor integrals can be represented in terms given in Appendix A. Those results are given in Appendix B.

2 Vertex integral

2.1 Massless one-loop vertex integral in n dimensions

The tensor integral of a massless one-loop vertex with rank $M \leq 3$ in a space-time dimension of $n = 4 - 2\varepsilon_{\rm UV}$ can be written as

$$T^{(3)}_{\underbrace{\mu\ldots\nu}_{M}} = \left(\mu_{\rm R}^2\right)^{\varepsilon_{\rm UV}} \int \frac{{\rm d}^n k}{(2\pi)^n} \frac{k_\mu \ldots k_\nu}{D_1 D_2 D_3},$$

where

$$\begin{aligned} D_1 &= k^2 + {\rm i}0, \\ D_2 &= (k+p_2)^2 + {\rm i}0, \\ D_3 &= (k+p_2+p_3)^2 + {\rm i}0, \end{aligned}$$

and p_i is the four momentum of an ith external particle (incoming), k_μ the loop momentum, and $\mu_{\rm R}$ the renormalization energy scale. An infinitesimal imaginary part (i0) is included to obtain analyticity of the integral $T^{(3)}_{\mu\ldots\nu}$. Momentum integration can be performed using Feynman's parameterization, which combines propagators. After momentum integration, an ultraviolet pole is subtracted under some renormalization scheme. Then space-time dimension is replaced as $\varepsilon_{\rm UV} \to -\varepsilon_{\rm IR}$ to regulate an infrared pole. Finally, the tensor integral is expressed in the following form [12]:

$$T^{(3)}_{\mu\ldots\nu} = \sum_i C^i_{\mu\ldots\nu} J^i_3(p_1^2, p_2^2, p_3^2; n_x^{(i)}, n_y^{(i)}),$$

where

$$\begin{aligned} &J^i_3(p_1^2, p_2^2, p_3^2; n_x^{(i)}, n_y^{(i)}) \\ &= \frac{1}{(4\pi)^2} \frac{\varepsilon_{\rm IR}\Gamma(-\varepsilon_{\rm IR})}{(4\pi\mu_{\rm R}^2)^{\varepsilon_{\rm IR}}} \int_0^1 {\rm d}x \int_0^{1-x} {\rm d}y \frac{x^{n_x^{(i)}} y^{n_y^{(i)}}}{D^{1-\varepsilon_{\rm IR}}}, \\ &D = (p_1 x - p_2 y)^2 - \rho x y - p_1^2 x - p_2^2 y - {\rm i}0, \\ &\rho = p_3^2 - (p_1+p_2)^2. \end{aligned} \tag{2.1}$$

The masses of internal particles are assumed to be massless. The remaining task is to perform the parametric integration of J_3.

2.2 Two on-shell, one off-shell external legs

For the case of two on-shell and one off-shell external particles, we set $p_1^2 = p_2^2 = 0, p_3^2 \neq 0$ without any loss of generality. The integration can be done in a straightforward way:

$$\begin{aligned} J_3(0,0,p_3^2; n_x, n_y) &= \frac{1}{(4\pi)^2} \frac{\varepsilon_{\rm IR}\Gamma(-\varepsilon_{\rm IR})}{(4\pi\mu_{\rm R}^2)^{\varepsilon_{\rm IR}}} \\ &\times \int_0^1 {\rm d}x \int_0^{1-x} {\rm d}y \frac{x^{n_x} y^{n_y}}{(-p_3^2 xy - {\rm i}0)^{1-\varepsilon_{\rm IR}}} \\ &= \frac{\varepsilon_{\rm IR}\Gamma(-\varepsilon_{\rm IR})}{(4\pi)^2} \left(\frac{-\tilde{p_3}^2}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} \frac{1}{-p_3^2} \frac{B(n_x+\varepsilon_{\rm IR}, n_y+\varepsilon_{\rm IR})}{n_x+n_y+2\varepsilon_{\rm IR}}, \end{aligned} \tag{2.2}$$

where $\tilde{p_3}^2 = p_3^2 + {\rm i}0$, and $B(\cdot,\cdot)$ is a beta function. The infrared structure of the tensor integral can be obtained by expanding (2.2) with respect to $\varepsilon_{\rm IR}$. The results of expansions under the $\overline{\rm MS}$ scheme are shown in Appendix B. When both n_x and n_y are non-zero, there is no IR divergence as

$$J_3(0,0,p_3^2; n_x, n_y) \to \frac{1}{(4\pi)^2 p_3^2} \frac{(n_x-1)!(n_y-1)!}{(n_x+n_y)!} \quad (\varepsilon_{\rm IR} \to 0).$$

2.3 One on-shell, two off-shell external legs

In the case of one on-shell and two off-shell external particles, we set $p_1^2 = 0, p_2^2 \neq 0, p_3^2 \neq 0$. The integral J_3 becomes

$$\begin{aligned} J_3(0,p_2^2,p_3^2; n_x, n_y) &= \frac{1}{(4\pi)^2} \frac{\varepsilon_{\rm IR}\Gamma(-\varepsilon_{\rm IR})}{(4\pi\mu_{\rm R}^2)^{\varepsilon_{\rm IR}}} \\ &\times \int_0^1 {\rm d}x \int_0^{1-x} {\rm d}y \\ &\times \frac{x^{n_x} y^{n_y}}{((p_3^2-p_2^2)xy - p_2^2 y(1-y) - {\rm i}0)^{1-\varepsilon_{\rm IR}}} \\ &= \frac{\varepsilon_{\rm IR}\Gamma(-\varepsilon_{\rm IR})}{(4\pi)^2} \left(\frac{-\tilde{p_3}^2}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} \frac{1}{-p_3^2} \frac{B(n_x+\varepsilon_{\rm IR}, n_y+\varepsilon_{\rm IR})}{n_x+n_y+2\varepsilon_{\rm IR}} \\ &\times {}_2F_1\left(1, 1-\varepsilon_{\rm IR}, 2+n_x; \frac{p_3^2-p_2^2}{\tilde{p_3}^2}\right) \frac{n_x+\varepsilon_{\rm IR}}{n_x+1} \\ &= J_3(0,0,p_3^2; n_x, n_y) \mathcal{G}_{n_x}\left(\frac{p_3^2-p_2^2}{\tilde{p_3}^2}\right), \end{aligned} \tag{2.3}$$

where

$$\mathcal{G}_n(z) = \frac{n+\varepsilon_{\rm IR}}{n+1} {}_2F_1\left(1, 1-\varepsilon_{\rm IR}, 2+n; z\right), \tag{2.4}$$

and ${}_2F_1(\cdot,\cdot,\cdot;\cdot)$ is the hypergeometric function. A definition and some properties of the hypergeometric function and its numerical evaluation can be found in Appendix A. When $p_2^2 \to 0$, the hypergeometric function becomes

$${}_2F_1\left(1, 1-\varepsilon_{\rm IR}, 2+n_x; z\right) \to \frac{n_x+1}{n_x+\varepsilon_{\rm IR}} \quad (z \to 1).$$

Then, the result (2.3) agrees with (2.2) when $p_2^2 \to 0$. The infrared structure of the tensor integral can be obtained by expanding (2.3) with respect to $\varepsilon_{\rm IR}$. The results of expansions under the $\overline{\rm MS}$ scheme are shown in Appendix B.

3 Box integral

3.1 Massless one-loop box integral in n dimensions

Box integrations can be treated the same as in the vertex case. The tensor integral of a massless one-loop vertex with rank $M \leq 4$ in space-time dimensions with $n = 4 - 2\varepsilon_{\rm UV}$ can be written as

$$T^{(4)}_{\underbrace{\mu\ldots\nu}_{M}} = \left(\mu_{\rm R}^2\right)^{\varepsilon_{\rm UV}} \int \frac{{\rm d}^n k}{(2\pi)^n {\rm i}} \frac{k_\mu \ldots k_\nu}{D_1 D_2 D_3 D_4},$$

where

$$\begin{aligned} D_1 &= k^2 + {\rm i}0, \\ D_2 &= (k+p_1)^2 + {\rm i}0, \\ D_3 &= (k+p_1+p_2)^2 + {\rm i}0, \\ D_4 &= (k+p_1+p_2+p_3)^2 + {\rm i}0. \end{aligned}$$

After the same procedure as that used in vertex integration, we are left with following parametric integration:

$$\begin{aligned} J_4(s,t;p_1^2,p_2^2,p_3^2,p_4^2;n_x,n_y,n_z) &= \frac{\Gamma(2-\varepsilon_{\rm IR})}{(4\pi)^2\left(4\pi\mu_{\rm R}^2\right)^{\varepsilon_{\rm IR}}} \\ \times \int_0^1 {\rm d}x \int_0^{1-x} {\rm d}y \int_0^{1-x-y} {\rm d}z \frac{x^{n_x} y^{n_y} z^{n_z}}{D^{2-\varepsilon_{\rm IR}}}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} D &= -sxz - ty(1-x-y-z) - p_1^2 xy - p_2^2 yz \\ &\quad -p_3^2 z(1-x-y-z) - p_4^2 x(1-x-y-z) - {\rm i}0, \\ s &= (p_1+p_2)^2, \\ t &= (p_1+p_4)^2. \end{aligned}$$

3.2 Four on-shell external legs

When all external particles are on-shell (massless), $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$, the integral of (3.1) can be

$$\begin{aligned} J_4(s,t;0,0,0,0;n_x,n_y,n_z) &= \frac{\Gamma(2-\varepsilon_{\rm IR})}{(4\pi)^2\left(4\pi\mu_{\rm R}^2\right)^{\varepsilon_{\rm IR}}} \\ \times \int_0^1 {\rm d}x{\rm d}y{\rm d}z \frac{x^{n_x} y^{n_y} z^{n_z}}{(-xzs - y(1-x-y-z)t - {\rm i}0)^{2-\varepsilon_{\rm IR}}}. \end{aligned} \tag{3.2}$$

After applying the transformation

$$\begin{aligned} x &= rv, \\ y &= w(1-r), \\ z &= (1-r)(1-w), \end{aligned}$$

the integral becomes

$$\begin{aligned} &J_4(s,t;0,0,0,0;n_x,n_y,n_z) \\ &= \frac{\Gamma(2-\varepsilon_{\rm IR})}{(4\pi)^2\left(4\pi\mu_{\rm R}^2\right)^{\varepsilon_{\rm IR}}} \int_0^1 {\rm d}r r^{-1+n_x+\varepsilon_{\rm IR}} (1-r)^{-1+n_y+n_z+\varepsilon_{\rm IR}} \\ &\quad \times \int_0^1 {\rm d}v \int_0^1 {\rm d}w \frac{w^{n_y}(1-w)^{n_z} v^{n_x}}{(-sv(1-w) - t(1-v)w - {\rm i}0)^{2-\varepsilon_{\rm IR}}} \\ &= \frac{\Gamma(2-\varepsilon_{\rm IR})}{(4\pi)^2} \left(4\pi\mu_{\rm R}^2\right)^{-\varepsilon_{\rm IR}} B(n_x+\varepsilon_{\rm IR}, n_y+n_z+\varepsilon_{\rm IR}) \\ &\quad \times \int_0^1 {\rm d}v \int_0^1 {\rm d}w \frac{w^{n_y}(1-w)^{n_z} v^{n_x}}{(-sv(1-w) - t(1-v)w - {\rm i}0)^{2-\varepsilon_{\rm IR}}}. \end{aligned}$$

The r-integral just gives the beta function. Then, the v-integral can be done:

$$\begin{aligned} I_v &\equiv \int_0^1 {\rm d}v \frac{v^{n_x}}{\left[(-s+sw+tw)\,v - tw - {\rm i}0\right]^{2-\varepsilon_{\rm IR}}} \\ &= \frac{\left(-\tilde{t}\right)^{\varepsilon_{\rm IR}} w^{\varepsilon_{\rm IR}-2}}{t^2(1+n_x)} {}_2F_1\left(2-\varepsilon_{\rm IR}, 1+n_x, 2+n_x, -\tilde{\xi}_w\right), \end{aligned}$$

where

$$\begin{aligned} \tilde{\xi}_w &= \frac{\tilde{u}}{\tilde{t}} + \frac{\tilde{s}}{\tilde{t}w}, \\ \tilde{s} &= s + {\rm i}0, \\ \tilde{t} &= t + {\rm i}0, \\ \tilde{u} &= u + {\rm i}0 = (p_1+p_3)^2 + {\rm i}0. \end{aligned}$$

Here, we use $s + t + u = \sum_i p_i^2 = 0$.

Further integration of the hypergeometric function is not straightforward. When a (or b) in the hypergeometric series ${}_2F_1(a,b,c;z)$ is a negative integer, the hypergeometric series is truncated at some point and becomes a polynomial. In order to express our target integrand as a polynomial, a transformation formula,

$$\begin{aligned} &{}_2F_1\left(a,b,c;z\right) \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} z^{-a} \\ &\quad \times {}_2F_1\left(a, a-c+1, a+b-c+1; 1-\frac{1}{z}\right) \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} z^{a-c} \\ &\quad \times {}_2F_1\left(c-a, 1-a, c-a-b+1; 1-\frac{1}{z}\right) \\ &\quad \left(|\arg z| < \pi, |\arg(1-z)| < \pi\right), \end{aligned}$$

is used. After this transformation, the hypergeometric function becomes

$$
\begin{aligned}
&{}_2F_1\left(2-\varepsilon_{\rm IR}, 1+n_x, 2+n_x, -\tilde{\xi}_w\right) \\
&= (n_x+1)!\Gamma(\varepsilon_{\rm IR}-1) \\
&\quad\times\left[\frac{1}{\Gamma(n_x+\varepsilon_{\rm IR})}\left(-\tilde{\xi}_w\right)^{-1-n_x}\right. \\
&\quad\left.+\frac{(1+\tilde{\xi}_w)^{\varepsilon_{\rm IR}-1}}{\tilde{\xi}_w}\sum_{l=0}^{n_x}\left(1+\frac{1}{\tilde{\xi}_w}\right)^l\frac{(-1)^l}{\Gamma(l+\varepsilon_{\rm IR})(n_x-l)!}\right].
\end{aligned}
\tag{3.3}
$$

Though the hypergeometric function in the LHS of (3.3) is finite when $n_x \geq 1$ and $\varepsilon_{\rm IR} \to 0$, the gamma function in the RHS has a $1/\varepsilon_{\rm IR}$ pole. We have confirmed that the terms in brackets on the RHS of (3.3) start $\mathcal{O}(\varepsilon_{\rm IR})$ when $n_x \geq 1$. Thus, there is no $1/\varepsilon_{\rm IR}$ pole, as expected.

Then, the remaining w-integral in J_4 becomes

$$
\begin{aligned}
&J_4(s,t;0,0,0,0;n_x,n_y,n_z) = \frac{-1}{(4\pi t)^2}\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} \\
&\times B(n_x+\varepsilon_{\rm IR}, n_y+n_z+\varepsilon_{\rm IR}) n_x!\Gamma(\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR}) \\
&\times\int_0^1 {\rm d}w w^{n_y-2+\varepsilon_{\rm IR}}(1-w)^{n_z} \\
&\times\left[\frac{1}{\Gamma(n_x+\varepsilon_{\rm IR})}\left(-\tilde{\xi}_w\right)^{-1-n_x}\right. \\
&\quad\left.+\frac{\left(1+\tilde{\xi}_w\right)^{\varepsilon_{\rm IR}-1}}{\tilde{\xi}_w}\sum_{l=0}^{n_x}\left(1+\frac{1}{\tilde{\xi}_w}\right)^l\frac{(-1)^l}{\Gamma(l+\varepsilon_{\rm IR})(n_x-l)!}\right].
\end{aligned}
$$

The w-integral can be solved in a term-by-term way for each power of l. The final form of J_4 is obtained to be

$$
\begin{aligned}
&J_4(s,t;0,0,0,0;n_x,n_y,n_z) = \frac{1}{(4\pi)^2 st} \\
&\times B(n_x+\varepsilon_{\rm IR}, n_y+n_z+\varepsilon_{\rm IR}) n_x!\Gamma(\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR}) \\
&\times\left[\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\left(\frac{-t}{s}\right)^{n_x}\frac{B(1+n_z, n_x+n_y+\varepsilon_{\rm IR})}{\Gamma(n_x+\varepsilon_{\rm IR})}\right. \\
&\times{}_2F_1\left(1+n_x, n_x+n_y+\varepsilon_{\rm IR},\right. \\
&\quad\left.1+n_x+n_y+n_z+\varepsilon_{\rm IR}, -\frac{\tilde{u}}{\tilde{s}}\right) \\
&+\left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\sum_{l=0}^{n_x}\left(\frac{-s}{t}\right)^l\frac{(-1)^l}{\Gamma(l+\varepsilon_{\rm IR})(n_x-l)!} \\
&\times B(1+n_y, l+n_z+\varepsilon_{\rm IR}) \\
&\times{}_2F_1\left(1+l, l+n_z+\varepsilon_{\rm IR},\right. \\
&\quad\left.\left.1+l+n_y+n_z+\varepsilon_{\rm IR}, -\frac{\tilde{u}}{\bar{t}}\right)\right],
\end{aligned}
\tag{3.4}
$$

where $\bar{t} = t - {\rm i}0$.

When the numerator of the integrand is unity, $n_x = n_y = n_z = 0$, the result of (3.4) is reduced to

$$
\begin{aligned}
&J_4(s,t;0,0,0,0;0,0,0) \\
&= \frac{1}{(4\pi)^2 st}\frac{B(\varepsilon_{\rm IR},\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR})}{\varepsilon_{\rm IR}} \\
&\quad\times\left[\left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}{}_2F_1\left(1,\varepsilon_{\rm IR},1+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\bar{t}}\right)\right. \\
&\quad\left.+\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}{}_2F_1\left(1,\varepsilon_{\rm IR},1+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{s}}\right)\right].
\end{aligned}
\tag{3.5}
$$

We have checked that this result agrees with the precedence result obtained by Duplanžić and Nižić [8] in both physical and unphysical regions of kinematical variables, s and t.

When $n_x = 0$, the result of (3.4) has a shorter expression without a fake pole, as follows:

$$
\begin{aligned}
&J_4(s,t;0,0,0,0;0,n_y,n_z) \\
&= \frac{1}{(4\pi)^2 st}B(\varepsilon_{\rm IR}, n_y+n_z+\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR}) \\
&\quad\times\left[\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}B(1+n_z, n_y+\varepsilon_{\rm IR})\right. \\
&\quad\times{}_2F_1\left(1, n_y+\varepsilon_{\rm IR}, 1+n_y+n_z+\varepsilon_{\rm IR}, -\frac{\tilde{u}}{\tilde{s}}\right) \\
&\quad+\left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}B(1+n_y, n_z+\varepsilon_{\rm IR}) \\
&\quad\left.\times{}_2F_1\left(1, n_z+\varepsilon_{\rm IR}, 1+n_y+n_z+\varepsilon_{\rm IR}, -\frac{\tilde{u}}{\bar{t}}\right)\right].
\end{aligned}
\tag{3.6}
$$

The infrared behavior of the loop integral can be obtained by expanding this formula with respect to $\varepsilon_{\rm IR}$, as shown in Appendix B.

In some cases with $n_x \neq 0$, we can avoid the fake pole by using the symmetry of the integrand. The basic integrand, (3.2), is symmetric under the exchange $x(n_x) \leftrightarrow z(n_z)$. Then, the result with $n_x \neq 0$ and $n_z = 0$ can be easily obtained as

$$
\begin{aligned}
&J_4(s,t;0,0,0,0;n_x,n_y,0) \\
&= \{J_4(s,t,0,0,0,0;0,n_y,n_z), n_z \to n_x\}
\end{aligned}
$$

for any values of n_y.

In the case of $n_x \neq 0, n_z \neq 0$, the integration has no IR divergence with any value of $n_y \geq 0$. However we cannot simply set $\varepsilon_{\rm IR}$ to be zero, because there is the fake pole. After expanding (3.4) with respect to $\varepsilon_{\rm IR}$, we confirmed that the $\varepsilon_{\rm IR}$ pole was canceled out. An explicit form of the Taylor expansion of the hypergeometric functions can be found in Appendix A.

3.3 One off-shell and three on-shell external legs

When one of four external particles is off-shell, we can set it to be p_1 without any loss of generality, $p_1^2 \neq 0, p_2^2 = p_3^2 = p_4^2 = 0$. Then, the integral of (3.1) can be written

$$\begin{aligned} &J_4(s,t;p_1^2,0,0,0;n_x,n_y,n_z) \\ &= \frac{\Gamma(2-\varepsilon_{\rm IR})}{(4\pi)^2\,(4\pi\mu_{\rm R}^2)^{\varepsilon_{\rm IR}}} \int_0^1 {\rm d}x \int_0^{1-x} {\rm d}y \int_0^{1-x-y} {\rm d}z \\ &\quad \times \frac{x^{n_x} y^{n_y} z^{n_z}}{(-xzs - y(1-x-y-z)t - p_1^2 xy - {\rm i}0)^{2-\varepsilon_{\rm IR}}}. \end{aligned} \tag{3.7}$$

After applying the same transformation as for the four-on-shell case, the integral becomes

$$\begin{aligned} &J_4(s,t;p_1^2,0,0,0;n_x,n_y,n_z) \\ &= \frac{\Gamma(2-\varepsilon_{\rm IR})}{(4\pi)^2\,(4\pi\mu_{\rm R}^2)^{\varepsilon_{\rm IR}}} B(n_x+\varepsilon_{\rm IR}, n_y+n_z+\varepsilon_{\rm IR}) \\ &\quad \times \int_0^1 {\rm d}v \int_0^1 {\rm d}w \\ &\quad \times \frac{w^{n_y}(1-w)^{n_z} v^{n_x}}{(-sv(1-w) - t(1-v)w - p_1^2 vw - {\rm i}0)^{2-\varepsilon_{\rm IR}}}. \end{aligned}$$

The v-integral can be done:

$$\begin{aligned} I_v &= \int_0^1 {\rm d}v \frac{v^{n_x}}{\left[(-s+sw+tw-p_1^2 w)\,v - tw - {\rm i}0\right]^{2-\varepsilon_{\rm IR}}} \\ &= \frac{\left(-\tilde{t}\right)^{\varepsilon_{\rm IR}} w^{\varepsilon_{\rm IR}-2}}{t^2(1+n_x)} {}_2F_1\left(2-\varepsilon_{\rm IR}, 1+n_x, 2+n_x, -\tilde{\xi}_w\right), \end{aligned}$$

where

$$\tilde{\xi}_w = \frac{\tilde{u}}{\tilde{t}} + \frac{\tilde{s}}{\tilde{t}w}.$$

Here, we use $s+t+u = p_1^2$. This result is the same as that in the four-on-shell case, except that $u = -s-t+p_1^2$ instead of $u = -s-t$. After making the same transformation as that for the four-on-shell case, the remaining w-integral can be expressed as

$$\begin{aligned} &J_4(s,t;p_1^2,0,0,0;n_x,n_y,n_z) = \frac{1}{(4\pi)^2 st} \\ &\times B(n_x+\varepsilon_{\rm IR}, n_y+n_z+\varepsilon_{\rm IR}) n_x! \Gamma(\varepsilon_{\rm IR}) \Gamma(1-\varepsilon_{\rm IR}) \\ &\times \left[\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} \left(\frac{-t}{s}\right)^{n_x} \frac{1}{\Gamma(n_x+\varepsilon_{\rm IR})} \mathcal{I}^{(1)} \right. \\ &\left. + \left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} \sum_{l=0}^{n_x} \frac{(-1)^l}{\Gamma(l+\varepsilon_{\rm IR})(n_x-l)!} \mathcal{I}_l^{(2)} \right], \end{aligned} \tag{3.8}$$

where

$$\mathcal{I}^{(1)} = \int_0^1 {\rm d}w w^{n_x+n_y-1+\varepsilon_{\rm IR}} (1-w)^{n_z} \left(1+\frac{\tilde{u}}{\tilde{s}}w\right)^{-1-n_x},$$

$$\begin{aligned} \mathcal{I}_l^{(2)} &= \frac{-t}{s} \int_0^1 {\rm d}w w^{n_y} (1-w)^{n_z} \\ &\quad \times \left(1+\frac{\tilde{u}}{\tilde{s}}w\right)^{-1-l} \left(1+\frac{\tilde{t}+\tilde{u}}{\tilde{s}}w\right)^{l-1+\varepsilon_{\rm IR}}. \end{aligned}$$

The first integration, $\mathcal{I}^{(1)}$, can be done:

$$\begin{aligned} \mathcal{I}^{(1)} &= B(1+n_z, n_x+n_y+\varepsilon_{\rm IR}) \\ &\quad \times {}_2F_1\Bigg(1+n_x, n_x+n_y+\varepsilon_{\rm IR}, \\ &\qquad 1+n_x+n_y+n_z+\varepsilon_{\rm IR}, -\frac{\tilde{u}}{\tilde{s}}\Bigg). \end{aligned} \tag{3.9}$$

For the second integration, $\mathcal{I}_l^{(2)}$, we used the following binomial expansion:

$$\begin{aligned} &(1-w)^{n_z} w^{n_y} = \sum_{k_1=0}^{n_z} {}_{n_z}C_{k_1} (-1)^{k_1} w^{n_y+k_1}, \\ &w^{n_y+k_1} = \left(\frac{s}{t+u}\right)^{n_y+k_1} \left(\left(1+\frac{\tilde{t}+\tilde{u}}{\tilde{s}}w\right)-1\right)^{n_y+k_1} \\ &= \left(\frac{s}{p_1^2-s}\right)^{n_y+k_1} \\ &\quad \times \sum_{k_2=0}^{n_y+k_1} {}_{n_y+k_1}C_{k_2} (-1)^{n_y+k_1-k_2} \left(1+\frac{\tilde{t}+\tilde{u}}{\tilde{s}}w\right)^{k_2}, \end{aligned}$$

where ${}_mC_n$ is the combinatorial defined as

$${}_mC_n \equiv \frac{m!}{n!(m-n)!}. \tag{3.10}$$

Then the second integration, $\mathcal{I}_l^{(2)}$, can be done:

$$\begin{aligned} &\mathcal{I}_l^{(2)} \\ &= \sum_{k_1=0}^{n_z} {}_{n_z}C_{k_1} \left(\frac{s}{p_1^2-s}\right)^{n_y+k_1} \\ &\quad \times \sum_{k_2=0}^{n_y+k_1} {}_{n_y+k_1}C_{k_2} (-1)^{n_y+k_2} \left(\frac{-t}{s}\right) \\ &\quad \times \int_0^1 {\rm d}w \left(1+\frac{\tilde{u}}{\tilde{s}}w\right)^{-(l+1)} \left(1+\frac{\tilde{t}+\tilde{u}}{\tilde{s}}w\right)^{k_2+l-1+\varepsilon_{\rm IR}} \\ &= \sum_{k_1=0}^{n_z} \sum_{k_2=0}^{n_y+k_1} {}_{n_z}C_{k_1\,n_y+k_1} C_{k_2} (-1)^{k_1+k_2} \left(\frac{s}{s-p_1^2}\right)^{n_y+k_1} \\ &\quad \times \frac{1}{l+k_2+\varepsilon_{\rm IR}} \left(1+\frac{u}{t}\right)^l \\ &\quad \times \left[{}_2F_1\left(1+l, l+k_2+\varepsilon_{\rm IR}, 1+l+k_2+\varepsilon_{\rm IR}, -\frac{\tilde{u}}{\tilde{t}}\right)\right. \end{aligned}$$

$$
\left.
-\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)^{l+k_2+\varepsilon_{\rm IR}}
\times {}_2F_1\left(1+l, l+k_2+\varepsilon_{\rm IR},\right.\right.
$$

$$
\left.\left.
1+l+k_2+\varepsilon_{\rm IR}, -\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right], \tag{3.11}
$$

where $\tilde{p_1}^2 = p_1^2 + {\rm i}0$. Here, the integral $J_4(s,t;p_1^2,0,0,0;n_x,n_y,n_z)$ can be successfully expressed by a finite number of hypergeometric functions. The infrared structure of the tensor integral can be obtained by expanding (3.8) with respect to $\varepsilon_{\rm IR}$. The results of expansions under the $\overline{\rm MS}$ scheme are given in Appendix B.

When the numerator of the integrand is unity, $n_x = n_y = n_z = 0$, the second integration (3.11) is reduced to

$$
\begin{aligned}
\mathcal{I}_0^{(2)} &= \frac{-t}{s}\int_0^1 {\rm d}w \left(1+\frac{\tilde{u}}{\tilde{s}}w\right)^{-1}\left(1+\frac{\tilde{t}+\tilde{u}}{\tilde{s}}w\right)^{\varepsilon_{\rm IR}-1} \\
&= \frac{1}{\varepsilon_{\rm IR}}\left[{}_2F_1\left(1,\varepsilon_{\rm IR},1+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{t}}\right)\right. \\
&\quad \left. -\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)^{\varepsilon_{\rm IR}} {}_2F_1\left(1,\varepsilon_{\rm IR},1+\varepsilon_{\rm IR},-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right].
\end{aligned} \tag{3.12}
$$

Then, (3.8) can be written as

$$
\begin{aligned}
&J_4(s,t;p_1^2,0,0,0;0,0,0) \\
&= \frac{1}{(4\pi)^2 st}\frac{B(\varepsilon_{\rm IR},\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR})}{\varepsilon_{\rm IR}} \\
&\quad \times\left[\left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} {}_2F_1\left(1,\varepsilon_{\rm IR},1+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{t}}\right)\right. \\
&\quad + \left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} {}_2F_1\left(1,\varepsilon_{\rm IR},1+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{s}}\right) \\
&\quad \left. -\left(\frac{-\tilde{p_1}^2}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} {}_2F_1\left(1,\varepsilon_{\rm IR},1+\varepsilon_{\rm IR},-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right] \\
&= J_4(s,t;0,0,0,0;0,0,0) \\
&\quad -\frac{1}{(4\pi)^2 st}\frac{B(\varepsilon_{\rm IR},\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR})}{\varepsilon_{\rm IR}}\left(\frac{-\tilde{p_1}^2}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} \\
&\quad \times {}_2F_1\left(1,\varepsilon_{\rm IR},1+\varepsilon_{\rm IR},-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\Bigg].
\end{aligned} \tag{3.13}
$$

We have again checked that this result agrees with the precedence result obtained by Duplanžić and Nižić [8] in both the physical and unphysical regions of the kinematical variables.

When $n_x = 0$, the result of (3.8) does not have a fake pole. The result is obtained to be

$$
\begin{aligned}
&J_4(s,t;p_1^2,0,0,0;0,n_y,n_z) \\
&= \frac{1}{(4\pi)^2 st}B(\varepsilon_{\rm IR},n_y+n_z+\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR}) \\
&\quad \times\left[\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\mathcal{I}^{(1)} + \left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\mathcal{I}_0^{(2)}\right],
\end{aligned} \tag{3.14}
$$

where

$$
\begin{aligned}
\mathcal{I}^{(1)} &= B(1+n_z,n_y+\varepsilon_{\rm IR}) \\
&\quad \times {}_2F_1\left(1,n_y+\varepsilon_{\rm IR},1+n_y+n_z+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{s}}\right).
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
\mathcal{I}_0^{(2)} &= \sum_{k_1=0}^{n_z}\sum_{k_2=0}^{n_y+k_1} {}_{n_z}C_{k_1}\,{}_{n_y+k_1}C_{k_2} \\
&\quad \times(-1)^{k_1+k_2}\left(\frac{s}{s-p_1^2}\right)^{n_y+k_1}\frac{1}{k_2+\varepsilon_{\rm IR}} \\
&\quad \times\left[{}_2F_1\left(1,k_2+\varepsilon_{\rm IR},1+k_2+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{t}}\right)\right. \\
&\quad \left. -\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)^{\varepsilon_{\rm IR}} {}_2F_1\left(1,k_2+\varepsilon_{\rm IR},1+k_2+\varepsilon_{\rm IR},-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right].
\end{aligned} \tag{3.16}
$$

In the case of $n_x \neq 0, n_z \neq 0$, the integration has no IR divergence when using any value of $n_y \geq 0$. However, we cannot simply set $\varepsilon_{\rm IR}$ to be zero again, because there is the fake pole. After expanding (3.8) with respect to $\varepsilon_{\rm IR}$, we confirmed numerically that the $\varepsilon_{\rm IR}$ pole was canceled out. The results of expansions under the $\overline{\rm MS}$ scheme are shown in Appendix B.

4 Numerical check of the results

The results of the vertex tensor integral are rather trivial. However, those of the box tensor integral are very complicated and highly non-trivial. We need some cross-checking of our results compared with other independent calculations. For the scalar integral of the one- or two-off-shell box integral, we can check our results numerically with those of the precedence calculation done by Duplanžić and Nižić [8]. In both in the physical and unphysical regions of the kinematical variables, s and t, it was confirmed that the results given in this report agree completely with those in [8].

The basic ingredients of a numerical calculation of the general case of the tensor integral are given in Appendix A. Those formulas are (A.3)–(A5) to evaluate the hypergeometric function and (A.10)–(A.21) to evaluate the generalized hypergeometric functions.

At first, the results of a numerical evaluation of the hypergeometric function of the type (A.4) are compared with those obtained from *Mathematica* [13] at several values of l, m, n and z at random. We confirmed that both results agree very well with each other to more than ten digits. For the generalized hypergeometric function, our recursion relation formulas, (A.10)–(A.21), were checked by comparing those of numerical integration of (A.9) using the numerical contour integral (NCI) method [14] developed by the author. The function $\tilde{F}_{l,m}^{(n)}(z)$, where $n = 1, 2$ and

Table 1. Numerical comparison of $\tilde{F}^{(1)}_{2,3}(z)$ between analytic formulas given in (A.11)–(A.15) and the numerical contour integral of (A.10)

z	real/imag.	analytic	NCI
$2000 + 10^{-15}\mathrm{i}$	real	$3.167042847 \times 10^{-7}$	$3.167042848 \times 10^{-7}$
	imag.	$-1.308996938 \times 10^{-7}$	$-1.308996939 \times 10^{-7}$
$-2000 + 10^{-15}\mathrm{i}$	real	$3.167042847 \times 10^{-7}$	$3.167042848 \times 10^{-7}$
	imag.	$\mathcal{O}(10^{-24})$	$\mathcal{O}(10^{-18})$
$0.2 + 10^{-15}\mathrm{i}$	real	$3.723185361 \times 10^{-1}$	$3.723185362 \times 10^{-1}$
	imag.	$\mathcal{O}(10^{-16})$	$\mathcal{O}(10^{-12})$
$-0.2 + 10^{-15}\mathrm{i}$	real	$1.809694496 \times 10^{-1}$	$1.809694496 \times 10^{-1}$
	imag.	$\mathcal{O}(10^{-16})$	$\mathcal{O}(10^{-12})$

Table 2. Numerical comparison of $\tilde{F}^{(2)}_{2,3}(z)$ between analytic formulas given in (A.17)–(A.21) and the numerical contour integral of (A.16)

z	real/imag.	analytic	NCI
$2000 + 10^{-15}\mathrm{i}$	real	$1.024874615 \times 10^{-6}$	$1.024874616 \times 10^{-6}$
	imag.	$-9.949558053 \times 10^{-7}$	$-9.949558053 \times 10^{-7}$
$-2000 + 10^{-15}\mathrm{i}$	real	$1.230491374 \times 10^{-6}$	$1.230491374 \times 10^{-6}$
	imag.	$\mathcal{O}(10^{-24})$	$\mathcal{O}(10^{-17})$
$0.2 + 10^{-15}\mathrm{i}$	real	$1.624874690 \times 10^{-1}$	$1.624874690 \times 10^{-1}$
	imag.	$\mathcal{O}(10^{-15})$	$\mathcal{O}(10^{-13})$
$-0.2 + 10^{-15}\mathrm{i}$	real	$1.005561296 \times 10^{-1}$	$1.005561296 \times 10^{-1}$
	imag.	$\mathcal{O}(10^{-15})$	$\mathcal{O}(10^{-12})$

$l = 2, m = 3$, given in (A.9), was numerically evaluated at several values of z, as shown in Tables 1 and 2, which was compared with the NCI results. Both results gave very good agreement to about ten digits, as shown in tables. The imaginary part of the result must be zero, except the $z = 2000 + 0\mathrm{i}$ case in the table. It was also numerically confirmed very precisely.

The numerical results of the box tensor integral with zero and one off-shell external legs obtained using (B.53) and (B.79) are also compared with those using the NCI method, integrating (3.1) and (3.7) directly. Both results show very good agreement within about five digits, as shown in Tables 3 and 4.

5 Conclusions

The general formulas of the 3- and 4-point tensor integral were obtained directly without any reduction method to the scalar integrals. The IR behavior of the tensor integrals was clearly shown by expanding the results with respect to the extra space-time dimensions due to the dimensional regularization. All results were expressed by very short formulas in a suitable manner for a numerical calculation. The results of the scalar integral were compared with the precedence results, and showed complete agreement in both physical and unphysical regions of the kinematical variables. For the IR finite case, the analytic results were compared with the numerical contour integration and gave a consistent result within the numerical integration error.

Table 3. Numerical comparison of $J_4(s,t;0,0,0,0;n_x,n_y,n_z)$ between analytic formulas given in (B.53) and the numerical contour integral. Here, we set kinematical variables in the physical region at $s = 123$, $t = -200$ and $\mu_{\mathrm{R}} = 1$

n_x	n_y	n_z	real/imag.	analytic	NCI
1	2	3	real	-2.15298×10^{-9}	-2.15297×10^{-9}
			imag.	-2.78647×10^{-9}	-2.78650×10^{-9}
2	0	2	real	9.74570×10^{-9}	9.74572×10^{-9}
			imag.	-3.22229×10^{-8}	-3.22230×10^{-8}

Table 4. Numerical comparison of $J_4(s,t;p_1^2,0,0,0;n_x,n_y,n_z)$ between analytic formulas given in (B.79) and the numerical contour integral. Here, we set the kinematical variables in the physical region $s = 123$, $t = -200$ $p_1^2 = 80$ and $\mu_{\mathrm{R}} = 1$

n_x	n_y	n_z	real/imag.	analytic	NCI
1	2	3	real	-7.88683×10^{-10}	-7.88689×10^{-10}
			imag.	-1.95176×10^{-9}	-1.95176×10^{-9}
2	0	2	real	1.48133×10^{-8}	1.48133×10^{-8}
			imag.	-2.04318×10^{-8}	-2.04318×10^{-8}

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Appendix A: Numerical calculation of the hypergeometric function

In this appendix, the basic properties of the hypergeometric function [15] and their numerical evaluation are summarized. The $1/\varepsilon_{\rm IR}$ expansion of the hypergeometric functions appearing in the tensor integrals and their numerical evaluation are also shown.

A.1 Hypergeometric function

The Gauss-series representation of the hypergeometric function is

$$\begin{aligned} {}_2F_1(a,b,c;z) &= {}_2F_1(b,a,c;z) \\ &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \end{aligned} \tag{A.1}$$

where $(\cdot)_k$ is Pochhammer's symbol defined as

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

The Euler integral representation is

$$\begin{aligned} &{}_2F_1(a,b,c;z) \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \tau^{b-1}(1-\tau)^{c-b-1}(1-z\tau)^{-a}\mathrm{d}\tau \\ &\quad (\mathcal{R}c > \mathcal{R}b > 0). \end{aligned} \tag{A.2}$$

When a is a negative integer, such as $a = -m$, the Gauss series is truncated at $k = m$ and becomes a polynomial,

$$\begin{aligned} {}_2F_1(-m,b,c;z) &= \sum_{k=0}^{m} \frac{(-m)_k (b)_k}{(c)_k} \frac{z^k}{k!} \\ &= \sum_{k=0}^{m} \frac{(b)_k}{(c)_k} {}_mC_k (-z)^k, \end{aligned} \tag{A.3}$$

where ${}_mC_k$ is the combinatorial defined in (3.10). For numerical evaluations of tensor integrals, the following type of the hypergeometric function might be numerically calculated as

$$\begin{aligned} &{}_2F_1(l, m+1, n+m+2; z) \\ &= \frac{1}{B(m+1,n+1)} \int_0^1 \tau^m (1-\tau)^n (1-z\tau)^{-l}\mathrm{d}\tau \\ &= \sum_{k_1=0}^{n} \sum_{k_2=0}^{m+k_1} (-1)^{k_1+k_2} \frac{{}_nC_{k_1}\, {}_{m+k_1}C_{k_2}}{B(m+1,n+1)} \frac{1}{z^{m+k_1}} \\ &\quad \times \int_0^1 (1-z\tau)^{-l+k_2}\mathrm{d}\tau, \end{aligned} \tag{A.4}$$

where l, m, n are positive integers. Here, integration can be performed as

$$\begin{aligned} &\int_0^1 (1-z\tau)^{-l+k_2}\mathrm{d}\tau \\ &= \begin{cases} -\dfrac{\ln(1-z)}{z}, & k_2 - l + 1 = 0, \\ \dfrac{1}{k_2-l+1} \dfrac{(1-z)^{k_2-l+1}-1}{-z}, & k_2 - l + 1 \neq 0. \end{cases} \end{aligned} \tag{A.5}$$

This formulas can be used for a numerical evaluation of hypergeometric functions of this type.

A.2 Generalized hypergeometric function

For a Laurent expansion of the hypergeometric function with respect to $\varepsilon_{\rm IR}$, the following generalized hypergeometric function is necessary:

$$\begin{aligned} &{}_3F_2(\{a_1,a_2,a_3\},\{b_1,b_2\};z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n} \frac{z^n}{n!} \end{aligned} \tag{A.6}$$

$$\begin{aligned} &= \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(b_1-a_1)\Gamma(a_2)\Gamma(b_2-a_2)} \\ &\quad \times \int_0^1 \mathrm{d}\tau \int_0^\tau \mathrm{d}v \\ &\quad \times v^{a_1-b_2}(1-v)^{b_1-a_1-1}\tau^{a_2-1} \\ &\quad \times (v-\tau)^{b_2-a_2-1}(1-z\tau)^{-a_3}, \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} &{}_4F_3(\{a_1,a_2,a_3,a_4\},\{b_1,b_2,b_3\};z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n (a_4)_n}{(b_1)_n (b_2)_n (b_3)_n} \frac{z^n}{n!}. \end{aligned} \tag{A.8}$$

Those generalized hypergeometric functions appear in the following integral:

$$\tilde{F}^{(n)}_{j_1,j_2}(z) \equiv \frac{(-1)^n}{n!} \int_0^1 \mathrm{d}\tau \tau^{j_1-1}(1-z\tau)^{-(j_2+1)} \ln^n \tau, \tag{A.9}$$

where j_1 is a positive integer and j_2 is an integer (it can be negative).

When $n = 1$, the integral becomes

$$\begin{aligned} &\tilde{F}^{(1)}_{j_1,j_2}(z) \\ &= -\int_0^1 \mathrm{d}\tau \tau^{j_1-1}(1-z\tau)^{-(j_2+1)} \ln\tau \end{aligned} \tag{A.10}$$

$$= \frac{{}_3F_2(\{j_1, j_1, j_2+1\}, \{j_1+1, j_1+1\}; z)}{j_1^2}$$
$$= \sum_{k=0}^{\infty} \frac{z^k}{(j_1+k)^2(j_2+k+1)B(j_2+1, k+1)}.$$

When j_2 is a negative integer, such as $j_2 = -j \leq -1$, this function can be expressed by a polynomial,

$$\tilde{F}^{(1)}_{j_1,-j}(z) = \sum_{k=0}^{j-1} \frac{{}_{j-1}C_k(-z)^k}{(j_1+k)^2}. \tag{A.11}$$

When $J_1 = 1$ and $j_2 = 0, 1$, the integral can be performed easily as

$$\tilde{F}^{(1)}_{1,0}(z) = \frac{\mathrm{Li}_2(z)}{z}, \tag{A.12}$$
$$\tilde{F}^{(1)}_{1,1}(z) = -\frac{\ln(1-z)}{z}. \tag{A.13}$$

When $j_1 = 1$ and $j_2 \geq 1$, we can use the following recursion relation:

$$\tilde{F}^{(1)}_{1,j_2+1}(z) = \frac{j_2}{j_2+1}\tilde{F}^{(1)}_{1,j_2} + \frac{(1-z)^{-j_2}-1}{j_2(j_2+1)z}. \tag{A.14}$$

For the general case, the function $\tilde{F}^{(1)}_{j_1,j_2}(z)$ can be obtained using $\tilde{F}^{(1)}_{1,\cdot}$,

$$\tilde{F}^{(1)}_{j_1,j_2}(z) = \frac{1}{z^{j_1-1}} \sum_{k=0}^{j_1-1} (-1)^k {}_{j_1-1}C_k \tilde{F}^{(1)}_{1,j_2-k}(z). \tag{A.15}$$

When $n = 2$, the integral becomes

$$\tilde{F}^{(2)}_{j_1,j_2}(z)$$
$$= \frac{1}{2}\int_0^1 \mathrm{d}\tau \tau^{j_1-1}(1-z\tau)^{-(j_2+1)} \ln^2\tau \tag{A.16}$$
$$= \frac{{}_4F_3(\{j_1, j_1, j_1, j_2+1\}, \{j_1+1, j_1+1, j_1+1\}; z)}{j_1^3}$$
$$= \sum_{k=0}^{\infty} \frac{z^k}{(j_1+k)^3(j_2+k+1)B(j_2+1, k+1)}.$$

When $j_2 = -j \leq -1$, it is also represented by a polynomial,

$$\tilde{F}^{(2)}_{j_1,-j}(z) = \sum_{k=0}^{j-1} \frac{{}_{j-1}C_k(-z)^k}{(j_1+k)^3}. \tag{A.17}$$

When $J_1 = 1$ and $j_2 = 0, 1$, the integral can be performed easily as follows:

$$\tilde{F}^{(2)}_{1,0}(z) = \frac{\mathrm{Li}_3(z)}{z}, \tag{A.18}$$
$$\tilde{F}^{(2)}_{1,1}(z) = \frac{\mathrm{Li}_2(z)}{z}. \tag{A.19}$$

When $j_1 = 1$ and $j_2 \geq 1$, we can use the following recursion relation:

$$(j_2+1)\tilde{F}^{(2)}_{1,j_2+1}(z) - j_2\tilde{F}^{(2)}_{1,j_2}(z) - \tilde{F}^{(1)}_{1,j_2}(z) = 0. \tag{A.20}$$

For the general case, the function $\tilde{F}^{(2)}_{j_1,j_2}(z)$ can be obtained using $\tilde{F}^{(2)}_{1,\cdot}(z)$,

$$\tilde{F}^{(2)}_{j_1,j_2}(z) = \frac{1}{z^{j_1-1}} \sum_{k=0}^{j_1-1} (-1)^k {}_{j_1-1}C_k \tilde{F}^{(2)}_{1,j_2-k}(z). \tag{A.21}$$

A.3 $1/\varepsilon_{\mathrm{IR}}$ expansion of a hypergeometric function

In the general form of the tensor integrals, the following type of integral appears:

$$\mathcal{I}_{l,m,n} \equiv \int_0^1 \tau^{l+n-1+\varepsilon_{\mathrm{IR}}}(1-\tau)^m(1-z\tau)^{-(l+1)}\mathrm{d}\tau,$$
$$= B(1+m, l+n+\varepsilon_{\mathrm{IR}}) \tag{A.22}$$
$$\times {}_2F_1\left(1+l, l+n+\varepsilon_{\mathrm{IR}}, 1+l+n+m+\varepsilon_{\mathrm{IR}}, z\right).$$

In order to show the IR structure of the function, a series expansion of the function $\mathcal{I}_{l,m,n}$ with respect to $\varepsilon_{\mathrm{IR}}$ might be considered,

$$\mathcal{I}_{l,m,n} = \sum_{j=-1}^{\infty} \mathcal{F}^{(j)}_{l,m,n}(z)\varepsilon^j_{\mathrm{IR}}, \tag{A.23}$$

where l, m, n are non-negative integers. When $l = m = n = 0$, the Laurent expansion of the function can be

$$\mathcal{I}_{0,0,0} = \int_0^1 \frac{(1-z\tau)^{-1}}{\tau^{1-\varepsilon_{\mathrm{IR}}}}\mathrm{d}\tau$$
$$= B(1, \varepsilon_{\mathrm{IR}}){}_2F_1\left(1, \varepsilon_{\mathrm{IR}}, 1+\varepsilon_{\mathrm{IR}}, z\right),$$
$$= \int_0^1 \mathrm{d}\tau \left[\frac{\delta(\tau)}{\varepsilon_{\mathrm{IR}}}(1-z\tau)^{-1} + \frac{(1-z\tau)^{-1}-1}{\tau}\right.$$
$$\left. + \varepsilon_{\mathrm{IR}}\frac{\ln\tau}{\tau}\left((1-z\tau)^{-1}-1\right) + \mathcal{O}\left(\varepsilon^2_{\mathrm{IR}}\right)\right] \tag{A.24}$$
$$= \frac{1}{\varepsilon_{\mathrm{IR}}} - \ln(1-z) - \mathrm{Li}(z)\varepsilon_{\mathrm{IR}} + \mathcal{O}\left(\varepsilon^2_{\mathrm{IR}}\right).$$

Then the first three terms of (A.23) are

$$\mathcal{F}^{(-1)}_{0,0,0}(z) = 1, \tag{A.25}$$
$$\mathcal{F}^{(0)}_{0,0,0}(z) = -\ln(1-z), \tag{A.26}$$
$$\mathcal{F}^{(1)}_{0,0,0}(z) = -\mathrm{Li}(z). \tag{A.27}$$

When $l = m = 0, n \neq 0$, the integral becomes

$$\mathcal{I}_{0,0,n} = \int_0^1 \tau^{n-1+\varepsilon_{\mathrm{IR}}}(1-z\tau)^{-1}\mathrm{d}\tau. \tag{A.28}$$

Then, the first three terms of (A.23) are

$$\mathcal{F}^{(0)}_{0,0,n}(z) = \frac{{}_2F_1(1,n,1+n;z)}{n}, \tag{A.29}$$

$$\mathcal{F}^{(1)}_{0,0,n}(z) = -\tilde{F}^{(1)}_{n,0}(z), \tag{A.30}$$

$$\mathcal{F}^{(2)}_{0,0,n}(z) = \tilde{F}^{(2)}_{n,0}(z). \tag{A.31}$$

When $l=0, m\neq 0, n=0$, the integral becomes

$$\begin{aligned}\mathcal{I}_{0,m,0} &= \int_0^1 \tau^{-1+\varepsilon_{\rm IR}}(1-\tau)^m(1-z\tau)^{-1}\mathrm{d}\tau \\ &= \sum_{k=0}^m {}_mC_k(-1)^k \int_0^1 \tau^{k-1+\varepsilon_{\rm IR}}(1-z\tau)^{-1}\mathrm{d}\tau \\ &= \sum_{k=0}^m {}_mC_k(-1)^k \mathcal{I}_{0,0,k}.\end{aligned} \tag{A.32}$$

Then, the first three terms of (A.23) are

$$\mathcal{F}^{(-1)}_{0,m,0}(z) = \mathcal{F}^{(-1)}_{0,0,0}(z), \tag{A.33}$$

$$\mathcal{F}^{(0)}_{0,m,0}(z) = \sum_{k=0}^m {}_mC_k(-1)^k \mathcal{F}^{(0)}_{0,0,k}(z), \tag{A.34}$$

$$\mathcal{F}^{(1)}_{0,m,0}(z) = \sum_{k=0}^m {}_mC_k(-1)^k \mathcal{F}^{(1)}_{0,0,k}(z). \tag{A.35}$$

When $l=0, m\neq 0, n\neq 0$, the integral becomes

$$\begin{aligned}\mathcal{I}_{0,m,n} &= \int_0^1 \tau^{n-1+\varepsilon_{\rm IR}}(1-\tau)^m(1-z\tau)^{-1}\mathrm{d}\tau \\ &= \sum_{k=0}^m {}_mC_k(-1)^k \int_0^1 \tau^{n+k-1+\varepsilon_{\rm IR}}(1-z\tau)^{-1}\mathrm{d}\tau \\ &= \sum_{k=0}^m {}_mC_k(-1)^k \mathcal{I}_{0,0,n+k}.\end{aligned} \tag{A.36}$$

Then, the first three terms of (A.23) are

$$\mathcal{F}^{(0)}_{0,m,n}(z) = B(1+m,n){}_2F_1(1,n,1+n+m;z), \tag{A.37}$$

$$\mathcal{F}^{(1)}_{0,m,n}(z) = \sum_{k=0}^m {}_mC_k(-1)^{k+1} \tilde{F}^{(1)}_{n+k,0}(z), \tag{A.38}$$

$$\mathcal{F}^{(2)}_{0,m,n}(z) = \sum_{k=0}^m {}_mC_k(-1)^{k} \tilde{F}^{(2)}_{n+k,0}(z). \tag{A.39}$$

When $l\neq 0, m=0, n=0$, the integral becomes

$$\mathcal{I}_{l,0,0} = \int_0^1 \tau^{l-1+\varepsilon_{\rm IR}}(1-z\tau)^{-(l+1)}\mathrm{d}\tau. \tag{A.40}$$

Then, the first three terms of (A.23) are

$$\mathcal{F}^{(0)}_{l,0,0}(z) = \frac{(1-z)^{-l}}{l}, \tag{A.41}$$

$$\mathcal{F}^{(1)}_{l,0,0}(z) = -\frac{{}_2F_1(l,l,l+1;z)}{l^2}, \tag{A.42}$$

$$\mathcal{F}^{(2)}_{l,0,0}(z) = \tilde{F}^{(2)}_{l,l}(z). \tag{A.43}$$

When $l\neq 0, m=0, n\neq 0$, the integral becomes

$$\mathcal{I}_{l,0,n} = \int_0^1 \tau^{l+n-1+\varepsilon_{\rm IR}}(1-z\tau)^{-(l+1)}\mathrm{d}\tau. \tag{A.44}$$

Then, the first three terms of (A.23) are

$$\mathcal{F}^{(0)}_{l,0,n}(z) = \frac{{}_2F_1(1+l,l+n,1+l+n;z)}{l+n}, \tag{A.45}$$

$$\mathcal{F}^{(1)}_{l,0,n}(z) = -\tilde{F}^{(1)}_{l+n,l}(z), \tag{A.46}$$

$$\mathcal{F}^{(2)}_{l,0,n}(z) = \tilde{F}^{(2)}_{l+n,l}(z). \tag{A.47}$$

When $l\neq 0, m\neq 0, n=0$, the integral becomes

$$\mathcal{I}_{l,m,0} = \int_0^1 \tau^{l-1+\varepsilon_{\rm IR}}(1-\tau)^m(1-z\tau)^{-(l+1)}\mathrm{d}\tau. \tag{A.48}$$

Then, the first three terms of (A.23) are

$$\mathcal{F}^{(0)}_{l,m,0}(z) = B(l,1+m){}_2F_1(1+l,l,1+l+m;z), \tag{A.49}$$

$$\mathcal{F}^{(1)}_{l,m,0}(z) = \sum_{k=0}^m {}_mC_k(-1)^{k+1} \tilde{F}^{(1)}_{l+k,l}(z), \tag{A.50}$$

$$\mathcal{F}^{(2)}_{l,m,0}(z) = \sum_{k=0}^m {}_mC_k(-1)^{k} \tilde{F}^{(2)}_{l+k,l}(z). \tag{A.51}$$

When $l\neq 0, m\neq 0, n\neq 0$, the integral becomes

$$\mathcal{I}_{l,m,n} = \int_0^1 \tau^{l+n-1+\varepsilon_{\rm IR}}(1-\tau)^m(1-z\tau)^{-(l+1)}\mathrm{d}\tau. \tag{A.52}$$

Then, the first three terms of (A.23) are

$$\begin{aligned}\mathcal{F}^{(0)}_{l,m,n} &= B(l+n,1+m) \\ &\quad \times {}_2F_1(1+l,l+n,1+l+m+n;z),\end{aligned} \tag{A.53}$$

$$\mathcal{F}^{(1)}_{l,m,n} = \sum_{k=0}^m {}_mC_k(-1)^{k+1} \tilde{F}^{(1)}_{l+n+k,l}(z), \tag{A.54}$$

$$\mathcal{F}^{(2)}_{l,m,n} = \sum_{k=0}^m {}_mC_k(-1)^{k} \tilde{F}^{(2)}_{l+n+k,l}(z). \tag{A.55}$$

Appendix B: $1/\varepsilon_{\rm IR}$ expansion of tensor integrations with the $\overline{\rm MS}$ scheme

We give $1/\varepsilon_{\rm IR}$ expansions of tensor integrals under the $\overline{\rm MS}$ scheme in this appendix. The $\overline{\rm MS}$ scheme is realized with the following replacement of the renormalization energy scale:

$$\mu_{\rm R}^2 \to \mu_{\rm R}^2 \frac{e^{\gamma_{\rm E}}}{4\pi},$$

where $\gamma_{\rm E}$ is Euler's constant.

B.1 Vertex with one off-shell external line

The general result of the vertex tensor integral is shown in (2.2). The IR structure of this function can be obtained by a Laurent expansion of the beta function for the IR singular case.

B.1.1 $n_x = n_y = 0$

When $n_x = n_y = 0$, the tensor integral becomes

$$J_3(0,0,p_3^2;0,0) = \frac{\Gamma(-\varepsilon_{\rm IR})}{(4\pi)^2}\left(\frac{-\tilde{p_3}^2}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\frac{1}{-p_3^2}\frac{B(\varepsilon_{\rm IR},\varepsilon_{\rm IR})}{2}. \quad \text{(B.1)}$$

After the $\varepsilon_{\rm IR}$ expansion with the $\overline{\rm MS}$ scheme,

$$J_3(0,0,p_3^2;0,0) \to \frac{1}{(4\pi)^2 p_3^2}\left[\frac{\mathcal{C}_{31}^{(-2)}}{\varepsilon_{\rm IR}^2}+\frac{\mathcal{C}_{31}^{(-1)}}{\varepsilon_{\rm IR}}+\mathcal{C}_{31}^{(0)}+O(\varepsilon_{\rm IR})\right], \quad \text{(B.2)}$$

where

$$\mathcal{C}_{31}^{(-2)} = 1, \quad \text{(B.3)}$$

$$\mathcal{C}_{31}^{(-1)} = \ln\left(\frac{-p_3^2}{\mu_{\rm R}^2}\right), \quad \text{(B.4)}$$

$$\mathcal{C}_{31}^{(0)} = -\frac{\pi^2}{12}+\frac{1}{2}\ln^2\left(\frac{-p_3^2}{\mu_{\rm R}^2}\right). \quad \text{(B.5)}$$

B.1.2 $n_x + n_y \neq 0$

The general result of the vertex tensor integral, (2.2), is symmetric under n_x and n_y exchanges. When one of n_x or n_y is non-zero, the tensor integral becomes

$$J_3(0,0,p_3^2;n,0) = J_3(0,0,p_3^2;0,n) = \frac{\varepsilon_{\rm IR}\Gamma(-\varepsilon_{\rm IR})}{(4\pi)^2}\left(\frac{-\tilde{p_3}^2}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\frac{1}{-p_3^2}\frac{B(n+\varepsilon_{\rm IR},\varepsilon_{\rm IR})}{n+2\varepsilon_{\rm IR}}. \quad \text{(B.6)}$$

After an $\varepsilon_{\rm IR}$ expansion with the $\overline{\rm MS}$ scheme,

$$J_3(0,0,p_3^2;n,0) = J_3(0,0,p_3^2;0,n) \to \frac{1}{(4\pi)^2 p_3^2}\left[\frac{\mathcal{C}_{32}^{(-1)}}{\varepsilon_{\rm IR}}+\mathcal{C}_{32}^{(0)}+O(\varepsilon_{\rm IR})\right], \quad \text{(B.7)}$$

where

$$\mathcal{C}_{32}^{(-1)} = \frac{1}{n}, \quad \text{(B.8)}$$

$$\mathcal{C}_{32}^{(0)} = -\frac{2}{n^2}+\frac{1}{n}\left(\ln\left(\frac{-p_3^2}{\mu_{\rm R}^2}\right)-\mathcal{H}_{n-1}\right), \quad \text{(B.9)}$$

where $\mathcal{H}_m$ is the harmonic number, defined as

$$\mathcal{H}_m = \textstyle\sum_{j=1}^{m}\frac{1}{j},$$

B.1.3 $n_x \neq 0, n_y \neq 0$

In this case, because there is no IR divergence in (2.2), $\varepsilon_{\rm IR}$ can be set to zero,

$$J_3(0,0,p_3^2;n_x,n_y) \to \frac{1}{(4\pi)^2 p_3^2}\frac{(n_x-1)!(n_y-1)!}{(n_x+n_y)!}. \quad \text{(B.10)}$$

B.2 Vertex with two off-shell external lines

B.2.1 $n_x = n_y = 0$

The function $\mathcal{G}_0(z)$, defined by (2.4), can be expanded with respect to $\varepsilon_{\rm IR}$ around zero as

$$\mathcal{G}_0(z) = \mathcal{G}_0^{(1)}(z)\varepsilon_{\rm IR}+\mathcal{G}_0^{(2)}(z)\varepsilon_{\rm IR}^2+\mathcal{O}\left(\varepsilon_{\rm IR}^3\right), \quad \text{(B.11)}$$

where

$$\mathcal{G}_0^{(1)}(z) = \frac{\ln(1-z)}{z}, \quad \text{(B.12)}$$

$$\mathcal{G}_0^{(2)}(z) = \frac{\ln^2(1-z)}{2z}. \quad \text{(B.13)}$$

Then, the vertex integral $J_3(0,p_2^2,p_3^2;0,0)$ can be expressed after an $\varepsilon_{\rm IR}$ expansion with the $\overline{\rm MS}$ scheme as

$$J_3(0,p_2^2,p_3^2;0,0) \to \frac{1}{(4\pi)^2 p_3^2}\left[\frac{\mathcal{C}_{33}^{(-1)}}{\varepsilon_{\rm IR}}+\mathcal{C}_{33}^{(0)}+O(\varepsilon_{\rm IR})\right], \quad \text{(B.14)}$$

where

$$\mathcal{C}_{33}^{(-1)} = \mathcal{C}_{31}^{(-2)}\mathcal{G}_0^{(1)}\left(\frac{p_3^2-p_2^2}{\tilde{p_3}^2}\right), \quad \text{(B.15)}$$

$$\mathcal{C}_{33}^{(0)} = \mathcal{C}_{31}^{(-1)}\mathcal{G}_0^{(1)}\left(\frac{p_3^2-p_2^2}{\tilde{p_3}^2}\right) + \mathcal{C}_{31}^{(-2)}\mathcal{G}_0^{(2)}\left(\frac{p_3^2-p_2^2}{\tilde{p_3}^2}\right). \quad \text{(B.16)}$$

B.2.2 $n_x = 0, n_y \neq 0$

Because the vertex integral $J_3(0,p_2^2,p_3^2;0,n)$ is IR finite, we can set $\varepsilon_{\rm IR}$ to be zero:

$$J_3(0,p_2^2,p_3^2;0,n) = \frac{1}{(4\pi)^2 p_3^2}\mathcal{C}_{34}^{(0)}, \quad \text{(B.17)}$$

where

$$\mathcal{C}_{34}^{(0)} = \mathcal{C}_{32}^{(-1)}\mathcal{G}_0^{(1)}\left(\frac{p_3^2-p_2^2}{\tilde{p_3}^2}\right). \quad \text{(B.18)}$$

B.2.3 $n_x \neq 0, n_y = 0$

For non-zero values of n, $\mathcal{G}_n(z)$ can be expanded with respect to $\varepsilon_{\rm IR}$ around zero as

$$\mathcal{G}_n(z) = \mathcal{G}_n^{(0)}(z) + \mathcal{G}_n^{(1)}(z)\varepsilon_{\rm IR} + \mathcal{O}\left(\varepsilon_{\rm IR}^2\right), \quad \text{(B.19)}$$

where

$$\mathcal{G}_n^{(0)}(z) = \frac{n}{n+1}{}_2F_1(1,1,2+n;z), \quad \text{(B.20)}$$

$$\begin{aligned}\mathcal{G}_n^{(1)}(z) &= \frac{1}{n+1}{}_2F_1(1,1,2+n;z) - n\left(\frac{z-1}{z}\right)^n \frac{\ln^2(1-z)}{2z} \\ &+ \frac{n}{z^n}\sum_{k=1}^{n} {}_nC_k (z-1)^{n-k} \\ &\times \frac{(1-z)^k\left(1-k\ln(1-z)\right)-1}{k^2 z}. \end{aligned} \quad \text{(B.21)}$$

Then, the vertex integral $J_3(0,p_2^2,p_3^2;n,0)$ can be expressed after an $\varepsilon_{\rm IR}$ expansion with the $\overline{\rm MS}$ scheme as

$$\begin{aligned}&J_3(0,p_2^2,p_3^2;n,0) \\ &\to \frac{1}{(4\pi)^2 p_3^2}\left[\frac{\mathcal{C}_{35}^{(-1)}}{\varepsilon_{\rm IR}} + \mathcal{C}_{35}^{(0)} + O(\varepsilon_{\rm IR})\right], \end{aligned} \quad \text{(B.22)}$$

where

$$\mathcal{C}_{35}^{(-1)} = \mathcal{C}_{32}^{(-1)}\mathcal{G}_0^{(0)}\left(\frac{p_3^2-p_2^2}{\tilde{p_3}^2}\right), \quad \text{(B.23)}$$

$$\begin{aligned}\mathcal{C}_{35}^{(0)} &= \mathcal{C}_{32}^{(-1)}\mathcal{G}_0^{(1)}\left(\frac{p_3^2-p_2^2}{\tilde{p_3}^2}\right) \\ &+ \mathcal{C}_{31}^{(0)}\mathcal{G}_0^{(0)}\left(\frac{p_3^2-p_2^2}{\tilde{p_3}^2}\right). \end{aligned} \quad \text{(B.24)}$$

B.2.4 $n_x \neq 0, n_y \neq 0$

In this case, there is no IR divergence and $\varepsilon_{\rm IR}$ can be set to zero,

$$\begin{aligned}&J_3(0,p_2^2,p_3^2;n_x,n_y) \\ &= \frac{1}{(4\pi)^2 p_3^2}\frac{(n_x-1)!(n_y-1)!}{(n_x+n_y)!} \\ &\times {}_2F_1\left(1,1,2+n_x;\frac{p_3^2-p_2^2}{\tilde{p_3}^2}\right)\frac{n_x}{n_x+1}. \end{aligned} \quad \text{(B.25)}$$

B.3 Box integral with all on-shell external legs

The general result of the box tensor integral with all on-shell external legs is shown in (3.4). The IR structure of this function can be obtained by a Laurent expansion of the beta function for the IR singular case.

B.3.1 $n_x = n_y = n_z = 0$

When the numerator of the integrand is unity, $n_x = n_y = n_z = 0$, the box tensor integral result, (3.4), is reduced to

$$\begin{aligned}&J_4(s,t,0,0,0,0;0,0,0) \\ &= \frac{1}{(4\pi)^2 st}\frac{B(\varepsilon_{\rm IR},\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR})}{\varepsilon_{\rm IR}} \\ &\times\left[\left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} {}_2F_1\left(1,\varepsilon_{\rm IR},1+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{t}}\right)\right. \\ &\left.+\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} {}_2F_1\left(1,\varepsilon_{\rm IR},1+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{s}}\right)\right]. \end{aligned} \quad \text{(B.26)}$$

After using a Laurent expansion of the hypergeometric function of (A.24) the loop integral with the $\overline{\rm MS}$ scheme is

$$\begin{aligned}&J_4(s,t,0,0,0,0;0,0,0) \\ &\to \frac{1}{(4\pi)^2 s\ t}\left[\frac{\mathcal{C}_{41}^{(-2)}}{\varepsilon_{\rm IR}^2} + \frac{\mathcal{C}_{41}^{(-1)}}{\varepsilon_{\rm IR}} + \mathcal{C}_{41}^{(0)} + O(\varepsilon_{\rm IR})\right], \end{aligned} \quad \text{(B.27)}$$

$$\mathcal{C}_{41}^{(-2)} = 4, \quad \text{(B.28)}$$

$$\mathcal{C}_{41}^{(-1)} = 2\left[\ln\left(\frac{-\tilde{s}}{\mu_{\rm R}^2}\right) + \ln\left(\frac{-\tilde{t}}{\mu_{\rm R}^2}\right)\right], \quad \text{(B.29)}$$

$$\begin{aligned}&\mathcal{C}_{41}^{(0)} \\ &= -\frac{\pi^2}{3} - 2{\rm Li}_2\left(-\frac{\tilde{u}}{\tilde{s}}\right) - 2{\rm Li}_2\left(-\frac{\tilde{u}}{\tilde{t}}\right) + \ln^2\left(\frac{-\tilde{s}}{\mu_{\rm R}^2}\right) \\ &+ \ln^2\left(\frac{-\tilde{t}}{\mu_{\rm R}^2}\right) - 2\ln\left(\frac{-\tilde{s}}{\mu_{\rm R}^2}\right)\ln\left(1+\frac{\tilde{u}}{\tilde{t}}\right) \\ &-2\ln\left(\frac{-\tilde{t}}{\mu_{\rm R}^2}\right)\ln\left(1+\frac{\tilde{u}}{\tilde{s}}\right). \end{aligned} \quad \text{(B.30)}$$

B.3.2 $n_x = 0, n_y = 0, n_z \neq 0$

When $n_x = n_y = 0, n_z \neq 0$, the result is reduced to

$$\begin{aligned}&J_4(s,t;0,0,0,0;0,0,n_z) \\ &= \frac{B(\varepsilon_{\rm IR},n_z+\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR})}{(4\pi)^2 st} \\ &\times\left[\left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} B(1,n_z+\varepsilon_{\rm IR})\right. \\ &\times {}_2F_1\left(1,n_z+\varepsilon_{\rm IR},1+n_z+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{t}}\right) \\ &+\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} B(\varepsilon_{\rm IR},1+n_z) \\ &\left.\times {}_2F_1\left(1,\varepsilon_{\rm IR},1+n_z+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{s}}\right)\right]. \end{aligned} \quad \text{(B.31)}$$

Laurent expansions of beta and gamma function with the $\overline{\rm MS}$ scheme are obtained as follows:

$$B(n_1+\varepsilon_{\rm IR}, n_2+\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR})\left(\frac{-q^2}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} \tag{B.32}$$
$$\to \frac{\mathcal{A}^{(-1)}_{n_1,n_2}(q^2)}{\varepsilon_{\rm IR}} + \mathcal{A}^{(0)}_{n_1,n_2}(q^2) + \mathcal{A}^{(1)}_{n_1,n_2}(q^2)\varepsilon_{\rm IR},$$

where

$$\mathcal{A}^{(-1)}_{0,n}(q^2) = 1, \tag{B.33}$$
$$\mathcal{A}^{(0)}_{0,n}(q^2) = \ln\left(\frac{-q^2}{\mu_{\rm R}^2}\right) - \mathcal{H}_{n-1}, \tag{B.34}$$
$$\mathcal{A}^{(1)}_{0,n}(q^2) = \frac{1}{6}\left[3\mathcal{H}^2_{n-1} + \pi^2 + 3\ln\left(\frac{-q^2}{\mu_{\rm R}^2}\right)\left(\ln\left(\frac{-q^2}{\mu_{\rm R}^2}\right) - 2\mathcal{H}_n\right) - 9\psi^{(1)}(n)\right], \tag{B.35}$$

and

$$\mathcal{A}^{(-1)}_{n_1,n_2}(q^2) = 0, \tag{B.36}$$
$$\mathcal{A}^{(0)}_{n_1,n_2}(q^2) = B(n_1,n_2), \tag{B.37}$$
$$\mathcal{A}^{(1)}_{n_1,n_2}(q^2) = B(n_1,n_2) \times\left(\mathcal{H}_{n_1-1} + \mathcal{H}_{n_2} - 2\mathcal{H}_{n_1+n_2-1} + \ln\left(\frac{-q^2}{\mu_{\rm R}^2}\right)\right), \tag{B.38}$$

where n, n_1, n_2 are positive integers. Here, $\psi^{(1)}(z)$ is the first derivative of the digamma function, given by

$$\psi^{(1)}(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1}\frac{1}{k^2}.$$

Finally, the tensor integral can be obtained to be

$$J_4(s,t;0,0,0,0;0,0,n_z) \to \frac{1}{(4\pi)^2 st}\left[\frac{\mathcal{C}^{(-2)}_{42}}{\varepsilon^2_{\rm IR}} + \frac{\mathcal{C}^{(-1)}_{42}}{\varepsilon_{\rm IR}} + \mathcal{C}^{(0)}_{42} + O(\varepsilon_{\rm IR})\right], \tag{B.39}$$

where

$$\mathcal{C}^{(-2)}_{42} = \mathcal{A}^{(-1)}_{0,n_z}(\tilde{t})\mathcal{F}^{(-1)}_{0,n_z,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right), \tag{B.40}$$

$$\mathcal{C}^{(-1)}_{42} = \mathcal{A}^{(-1)}_{0,n_z}(\tilde{s})\mathcal{F}^{(0)}_{0,0,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right) + \mathcal{A}^{(-1)}_{0,n_z}(\tilde{t})\mathcal{F}^{(0)}_{0,n_z,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right) + \mathcal{A}^{(0)}_{0,n_z}(\tilde{t})\mathcal{F}^{(-1)}_{0,n_z,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right), \tag{B.41}$$

$$\mathcal{C}^{(0)}_{42} = \mathcal{A}^{(-1)}_{0,n_z}(\tilde{s})\mathcal{F}^{(1)}_{0,0,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right) + \mathcal{A}^{(0)}_{0,n_z}(\tilde{s})\mathcal{F}^{(0)}_{0,0,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right) + \mathcal{A}^{(-1)}_{0,n_z}(\tilde{t})\mathcal{F}^{(1)}_{0,n_z,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right) + \mathcal{A}^{(0)}_{0,n_z}(\tilde{t})\mathcal{F}^{(0)}_{0,n_z,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right) + \mathcal{A}^{(1)}_{0,n_z}(\tilde{t})\mathcal{F}^{(-1)}_{0,n_z,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right). \tag{B.42}$$

B.3.3 $n_x = 0, n_y \neq 0, n_z = 0$

When $n_x = 0$ and both n_y and n_z are non-zero, J_4 becomes

$$J_4(s,t;0,0,0,0;0,n_y,n_z) = \frac{1}{(4\pi)^2 st}B(\varepsilon_{\rm IR}, n_y+n_z+\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR}) \times\left[\left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} B(1+n_y, n_z+\varepsilon_{\rm IR}) \times {}_2F_1\left(1, n_z+\varepsilon_{\rm IR}, 1+n_y+n_z+\varepsilon_{\rm IR}, -\frac{\tilde{u}}{\tilde{t}}\right) + \left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}} B(1+n_z, n_y+\varepsilon_{\rm IR}) \times {}_2F_1\left(1, n_y+\varepsilon_{\rm IR}, 1+n_y+n_z+\varepsilon_{\rm IR}, -\frac{\tilde{u}}{\tilde{s}}\right)\right]. \tag{B.43}$$

One can see that $J_4(s,t;0,0,0,0;0,n_y,n_z)$ is symmetric under a simultaneous exchange of $t \leftrightarrow s, n_y \leftrightarrow n_z$. Then, for the case of $n_x = n_z = 0, n_y \neq 0$, the required integration can be obtained as

$$J_4(s,t;0,0,0;0,n_y,0) = \{J_4(s,t;0,0,0;0,0,n_z), n_z \to n_y, s \leftrightarrow t\}. \tag{B.44}$$

B.3.4 $n_x = 0, n_y \neq 0, n_z \neq 0$

Let us start from (B.43). There is no IR singular pole in the square bracket and the $\frac{1}{\varepsilon_{\rm IR}}$ pole in the Beta function in front of the square brackets. One thus needs to expand the terms in the square bracket up to $O(\varepsilon_{\rm IR})$.

Then, finally we obtain

$$J_4(s,t;0,0,0,0;0,n_y,n_z) = \frac{1}{(4\pi)^2 st}B(\varepsilon_{\rm IR}, n_y+n_z+\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR}) \times\left[\left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\left(\mathcal{F}^{(0)}_{0,n_y,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right) + \varepsilon_{\rm IR}\mathcal{F}^{(1)}_{0,n_y,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right)\right)\right. \tag{B.45}$$

$$+\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\left(\mathcal{F}^{(0)}_{0,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)+\varepsilon_{\rm IR}\mathcal{F}^{(1)}_{0,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\right)\Bigg].$$

After an $\varepsilon_{\rm IR}$ expansion with the $\overline{\rm MS}$ scheme, it is obtained as

$$J_4(s,t;0,0,0,0,0;0,n_y,n_z)\to\frac{1}{(4\pi)^2st}\left[\frac{\mathcal{C}^{(-1)}_{43}}{\varepsilon_{\rm IR}}+\mathcal{C}^{(0)}_{43}+O(\varepsilon_{\rm IR})\right],\quad\text{(B.46)}$$

where

$$\mathcal{C}^{(-1)}_{43}=\mathcal{F}^{(0)}_{0,n_y,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right)+\mathcal{F}^{(0)}_{0,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right),\quad\text{(B.47)}$$

$$\begin{aligned}\mathcal{C}^{(0)}_{43}=&\mathcal{F}^{(1)}_{0,n_y,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right)+\mathcal{F}^{(1)}_{0,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\&+\mathcal{F}^{(0)}_{0,n_y,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right)\left(\ln\left(-\frac{\tilde{s}}{\mu_{\rm R}^2}\right)-\mathcal{H}_{n_y+n_z-1}\right)\\&+\mathcal{F}^{(0)}_{0,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\left(\ln\left(-\frac{\tilde{t}}{\mu_{\rm R}^2}\right)-\mathcal{H}_{n_y+n_z-1}\right).\end{aligned}$$

B.3.5 $n_x\neq0,n_y=0,n_z=0$

The basic integrand, (3.2), is symmetric under the exchange $x(n_x)\leftrightarrow z(n_z)$. Then, the result can be easily obtained as

$$\begin{aligned}&J_4(s,t;0,0,0,0,0;n_x,0,0)\\&=\{J_4(s,t;0,0,0,0,0;0,0,n_z),n_z\to n_x\}.\end{aligned}\quad\text{(B.48)}$$

B.3.6 $n_x\neq0,n_y\neq0,n_z=0$

When $n_z=0$, the result of the integration, (3.4), can be written

$$\begin{aligned}&J_4(s,t;0,0,0,0,0;n_x,n_y,0)\\&=\frac{1}{(4\pi)^2st}B(n_x+\varepsilon_{\rm IR},n_y+\varepsilon_{\rm IR})n_x!\Gamma(\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR})\\&\times\Bigg[\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\left(\frac{-t}{s}\right)^{n_x}\frac{B(1+n_x+n_y+\varepsilon_{\rm IR})}{\Gamma(n_x+\varepsilon_{\rm IR})}\\&\times{}_2F_1\left(1+n_x,n_x+n_y+\varepsilon_{\rm IR},1+n_x+n_y+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{s}}\right)\\&+\left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\sum_{l=0}^{n_x}\left(\frac{-s}{t}\right)^l\frac{(-1)^l}{\Gamma(l+\varepsilon_{\rm IR})(n_x-l)!}\\&\times B(1+n_y,l+\varepsilon_{\rm IR})\\&\times{}_2F_1\left(1+l,l+\varepsilon_{\rm IR},1+l+n_y+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{t}}\right)\Bigg].\end{aligned}\quad\text{(B.49)}$$

After an $\varepsilon_{\rm IR}$ expansion with the $\overline{\rm MS}$ scheme, it is obtained as

$$J_4(s,t;0,0,0,0,0;n_x,n_y,0)\to\frac{1}{(4\pi)^2st}\left[\frac{\mathcal{C}^{(-1)}_{44}}{\varepsilon_{\rm IR}}+\mathcal{C}^{(0)}_{44}+O(\varepsilon_{\rm IR})\right],\quad\text{(B.50)}$$

where

$$\begin{aligned}\mathcal{C}^{(-1)}_{44}=&n_x\left(-\frac{t}{s}\right)^{n_x}\mathcal{A}^{(0)}_{n_x,n_y}(\tilde{t})\mathcal{F}^{(0)}_{n_x,0,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\&+\mathcal{A}^{(0)}_{n_x,n_y}(\tilde{s})\mathcal{F}^{(-1)}_{0,n_y,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)\\&+\sum_{l=1}^{n_x}l\left(\frac{s}{t}\right)^l{}_{n_x}C_l\mathcal{A}^{(0)}_{n_x,n_y}(\tilde{s})\mathcal{F}^{(0)}_{l,n_y,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right),\end{aligned}\quad\text{(B.51)}$$

$$\begin{aligned}\mathcal{C}^{(0)}_{44}=&n_x\left(-\frac{t}{s}\right)^{n_x}\\&\times\Bigg[\mathcal{A}^{(1)}_{n_x,n_y}(\tilde{t})\mathcal{F}^{(0)}_{n_x,0,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\&+\mathcal{A}^{(0)}_{n_x,n_y}(\tilde{t})\mathcal{F}^{(1)}_{n_x,0,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\&-\mathcal{A}^{(0)}_{n_x,n_y}(\tilde{t})\mathcal{F}^{(0)}_{n_x,0,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\mathcal{H}_{n_x-1}\Bigg]\\&+\mathcal{A}^{(1)}_{n_x,n_y}(\tilde{s})\mathcal{F}^{(-1)}_{0,n_y,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)+\mathcal{A}^{(0)}_{n_x,n_y}(\tilde{s})\mathcal{F}^{(0)}_{0,n_y,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)\\&+\sum_{l=1}^{n_x}l\left(\frac{s}{t}\right)^l{}_{n_x}C_l\Bigg[\mathcal{A}^{(1)}_{n_x,n_y}(\tilde{s})\mathcal{F}^{(0)}_{l,n_y,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)\\&+\mathcal{A}^{(0)}_{n_x,n_y}(\tilde{s})\left(\mathcal{F}^{(1)}_{l,n_y,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)-\mathcal{F}^{(0)}_{l,n_y,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)\mathcal{H}_{l-1}\right)\Bigg].\end{aligned}$$

B.3.7 $n_x\neq0$, $n_y=$ any value, $n_z\neq0$

In the case of $n_x\neq0,n_z\neq0$, the integration has no IR divergence with any value of $n_y\geq0$. The result is given by

$$J_4(s,t;0,0,0,0,0;n_x,n_y,n_z)\to\frac{1}{(4\pi)^2st}\left[\mathcal{C}^{(0)}_{45}+O(\varepsilon_{\rm IR})\right],\quad\text{(B.52)}$$

where

$$\begin{aligned}\mathcal{C}^{(0)}_{45}=&n_x\left(-\frac{t}{s}\right)^{n_x}\Bigg[\mathcal{A}^{(1)}_{n_x,n_y+n_z}(\tilde{t})\mathcal{F}^{(0)}_{n_x,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\&+\mathcal{A}^{(0)}_{n_x,n_y+n_z}(\tilde{t})\\&\times\left(\mathcal{F}^{(1)}_{n_x,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)-\mathcal{F}^{(0)}_{n_x,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\mathcal{H}_{n_x-1}\right)\Bigg]\\&+\mathcal{A}^{(0)}_{n_x,n_y+n_z}(\tilde{s})\mathcal{F}^{(0)}_{0,n_y,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right)\end{aligned}$$

$$+\sum_{l=1}^{n_x} l \left(\frac{s}{t}\right)^l {}_{n_x}C_l \Bigg[\mathcal{A}^{(1)}_{n_x,n_y+n_z}(\tilde{s})\mathcal{F}^{(0)}_{l,n_y,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right) \quad \text{(B.53)}$$

$$+\mathcal{A}^{(0)}_{n_x,n_y+n_z}(\tilde{s})\left(\mathcal{F}^{(1)}_{l,n_y,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right)-\mathcal{F}^{(0)}_{l,n_y,n_z}\left(-\frac{\tilde{u}}{\tilde{t}}\right)\mathcal{H}_{l-1}\right)\Bigg].$$

B.4 Box integral with one off-shell and three on-shell external legs

The general result of the box tensor integral with one off-shell and three on-shell external legs is shown in (3.8) with (3.9) and (3.11). The IR structure of this function can be obtained by the Laurent expansion of the beta function for the IR singular case.

B.4.1 $n_x = n_y = n_z = 0$

When the numerator of the integrand is unity, $n_x = n_y = n_z = 0$, the second integral $\mathcal{I}^{(2)}$ is reduced to

$$\begin{aligned}\mathcal{I}^{(2)} &= \int_0^1 \mathrm{d}w \left(1+\frac{\tilde{u}}{\tilde{s}}w\right)^{-1}\left(1+\frac{\tilde{t}+\tilde{u}}{\tilde{s}}w\right)^{\varepsilon_{\mathrm{IR}}-1} \quad \text{(B.54)}\\ &= \frac{-s}{t}\frac{1}{\varepsilon_{\mathrm{IR}}}\left[{}_2F_1\left(1,\varepsilon_{\mathrm{IR}},1+\varepsilon_{\mathrm{IR}},-\frac{\tilde{u}}{\tilde{t}}\right)\right.\\ &\quad \left.-\left(\frac{\tilde{p}_1^2}{\tilde{s}}\right)^{\varepsilon_{\mathrm{IR}}}{}_2F_1\left(1,\varepsilon_{\mathrm{IR}},1+\varepsilon_{\mathrm{IR}},-\frac{\tilde{u}\tilde{p}_1^2}{\tilde{t}\tilde{s}}\right)\right],\end{aligned}$$

where $\tilde{p}_1^2 = p_1^2 + \mathrm{i}0$. Then, the final result can be written as

$$\begin{aligned}&J_4(s,t;p_1^2,0,0,0;0,0,0)\\ &= \frac{1}{(4\pi)^2 st}\frac{B(\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{IR}})\Gamma(1-\varepsilon_{\mathrm{IR}})}{\varepsilon_{\mathrm{IR}}}\\ &\quad\times\left[\left(\frac{-\tilde{s}}{4\pi\mu_{\mathrm{R}}^2}\right)^{\varepsilon_{\mathrm{IR}}}{}_2F_1\left(1,\varepsilon_{\mathrm{IR}},1+\varepsilon_{\mathrm{IR}},-\frac{\tilde{u}}{\tilde{t}}\right)\right.\\ &\quad+\left(\frac{-\tilde{t}}{4\pi\mu_{\mathrm{R}}^2}\right)^{\varepsilon_{\mathrm{IR}}}{}_2F_1\left(1,\varepsilon_{\mathrm{IR}},1+\varepsilon_{\mathrm{IR}},-\frac{\tilde{u}}{\tilde{s}}\right)\\ &\quad\left.-\left(\frac{-\tilde{p}_1^2}{4\pi\mu_{\mathrm{R}}^2}\right)^{\varepsilon_{\mathrm{IR}}}{}_2F_1\left(1,\varepsilon_{\mathrm{IR}},1+\varepsilon_{\mathrm{IR}},-\frac{\tilde{u}\tilde{p}_1^2}{\tilde{t}\tilde{s}}\right)\right],\\ &= J_4(s,t;0,0,0,0;0,0,0)\\ &\quad-\frac{1}{(4\pi)^2 st}\frac{B(\varepsilon_{\mathrm{IR}},\varepsilon_{\mathrm{IR}})\Gamma(1-\varepsilon_{\mathrm{IR}})}{\varepsilon_{\mathrm{IR}}}\left(\frac{-\tilde{p}_1^2}{4\pi\mu_{\mathrm{R}}^2}\right)^{\varepsilon_{\mathrm{IR}}}\\ &\quad\times{}_2F_1\left(1,\varepsilon_{\mathrm{IR}},1+\varepsilon_{\mathrm{IR}},-\frac{\tilde{u}\tilde{p}_1^2}{\tilde{t}\tilde{s}}\right)\Bigg]. \quad \text{(B.55)}\end{aligned}$$

After an $\varepsilon_{\mathrm{IR}}$ expansion with the $\overline{\mathrm{MS}}$ scheme, it is obtained as

$$\begin{aligned}&J_4(s,t,p_1^2,0,0,0;0,0,0) \quad \text{(B.56)}\\ &\to \frac{1}{(4\pi)^2 st}\left[\frac{\mathcal{C}_{46}^{(-2)}}{\varepsilon_{\mathrm{IR}}^2}+\frac{\mathcal{C}_{46}^{(-1)}}{\varepsilon_{\mathrm{IR}}}+\mathcal{C}_{46}^{(0)}+O(\varepsilon_{\mathrm{IR}})\right],\end{aligned}$$

where

$$\mathcal{C}_{46}^{(-2)} = \mathcal{C}_{41}^{(-2)} - 2, \quad \text{(B.57)}$$

$$\mathcal{C}_{46}^{(-1)} = \mathcal{C}_{41}^{(-1)} - 2\left[\ln\left(\frac{-\tilde{p}_1^2}{\mu_{\mathrm{R}}^2}\right)-\ln\left(1+\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right], \quad \text{(B.58)}$$

$$\begin{aligned}\mathcal{C}_{46}^{(0)} &= \mathcal{C}_{41}^{(0)} + \left[\frac{\pi^2}{6}+2\mathrm{Li}_2\left(-\frac{\tilde{u}\tilde{p}_1^2}{\tilde{t}\tilde{s}}\right)-\ln^2\left(\frac{-\tilde{p}_1^2}{\mu_{\mathrm{R}}^2}\right)\right.\\ &\quad\left.+2\ln\left(\frac{-\tilde{p}_1^2}{\mu_{\mathrm{R}}^2}\right)\ln\left(1+\frac{\tilde{u}\tilde{p}_1^2}{\tilde{t}\tilde{s}}\right)\right]. \quad \text{(B.59)}\end{aligned}$$

B.4.2 $n_x = 0, n_y + n_z \geq 1$

When $n_x = 0$, and n_y, n_z have any value with $n_y + n_z \geq 1$, the final result can be written as

$$\begin{aligned}&J_4(s,t,p_1^2,0,0,0;0,n_y,n_z)\\ &= \frac{1}{(4\pi)^2 st}B(\varepsilon_{\mathrm{IR}},n_y+n_z+\varepsilon_{\mathrm{IR}})\Gamma(1-\varepsilon_{\mathrm{IR}})\\ &\quad\times\left[\left(\frac{-\tilde{t}}{4\pi\mu^2}\right)^{\varepsilon_{\mathrm{IR}}}\mathcal{I}^{(1)}+\left(\frac{-\tilde{s}}{4\pi\mu^2}\right)^{\varepsilon_{\mathrm{IR}}}\mathcal{I}^{(2)}\right]. \quad \text{(B.60)}\end{aligned}$$

The integrals $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ can be done easily:

$$\begin{aligned}\mathcal{I}^{(1)} &= B(n_y+\varepsilon_{\mathrm{IR}},1+n_z)\\ &\quad\times{}_2F_1\left(1,n_y+\varepsilon_{\mathrm{IR}},1+n_y+n_z+\varepsilon_{\mathrm{IR}},-\frac{\tilde{u}}{\tilde{s}}\right), \quad \text{(B.61)}\\ \mathcal{I}_0^{(2)} &= \sum_{k_1=0}^{n_z}\sum_{k_2=0}^{n_y+k_1}{}_{n_z}C_{k_1\,n_y+k_1}C_{k_2}(-1)^{k_1+k_2}\\ &\quad\times\left(\frac{s}{s-p_1^2}\right)^{n_y+k_1}\frac{1}{k_2+\varepsilon_{\mathrm{IR}}}\\ &\quad\times\left[{}_2F_1\left(1,k_2+\varepsilon_{\mathrm{IR}},1+k_2+\varepsilon_{\mathrm{IR}},-\frac{\tilde{u}}{\tilde{t}}\right)\right. \quad \text{(B.62)}\\ &\quad\left.-\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)^{\varepsilon_{\mathrm{IR}}}{}_2F_1\left(1,k_2+\varepsilon_{\mathrm{IR}},1+k_2+\varepsilon_{\mathrm{IR}},-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right].\end{aligned}$$

After an $\varepsilon_{\mathrm{IR}}$ expansion with the $\overline{\mathrm{MS}}$ scheme, it is obtained as

$$\begin{aligned}&J_4(s,t,p_1^2,0,0,0;0,n_y,n_z) \quad \text{(B.63)}\\ &\to \frac{1}{(4\pi)^2 st}\left[\frac{\mathcal{C}_{47}^{(-2)}}{\varepsilon_{\mathrm{IR}}^2}+\frac{\mathcal{C}_{47}^{(-1)}}{\varepsilon_{\mathrm{IR}}}+\mathcal{C}_{47}^{(0)}+O(\varepsilon_{\mathrm{IR}})\right],\end{aligned}$$

where

$$\mathcal{C}_{47}^{(-2)} = \mathcal{A}^{(-1)}_{0,n_y+n_z}(\tilde{t})\mathcal{F}^{(-1)}_{0,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right), \quad \text{(B.64)}$$

$\mathcal{C}_{47}^{(-1)}$

$$
\begin{aligned}
&= \mathcal{A}^{(-1)}_{0,n_y+n_z}(\tilde{t})\mathcal{F}^{(0)}_{0,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\
&\quad+\mathcal{A}^{(0)}_{0,n_y+n_z}(\tilde{t})\mathcal{F}^{(-1)}_{0,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\
&\quad+\sum_{k_1=0}^{n_z}\sum_{k_2=0}^{n_y+k_1} {}_{n_z}C_{k_1\,n_y+k_1}C_{k_2}(-1)^{k_1+k_2}\left(\frac{s}{s-p_1^2}\right)^{n_y+k_1}\\
&\quad\times\mathcal{A}^{(-1)}_{0,n_y+n_z}(\tilde{s})\\
&\quad\times\left[\mathcal{F}^{(0)}_{0,0,k_2}\left(-\frac{\tilde{u}}{\bar{t}}\right)\right.\\
&\quad-\left(\frac{p_1^2}{s}\right)^{k_2}\left(\mathcal{F}^{(0)}_{0,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right.\\
&\quad\left.\left.+\ln\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)\mathcal{F}^{(-1)}_{0,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right)\right],
\end{aligned}
\tag{B.65}
$$

$$
\begin{aligned}
&\mathcal{C}^{(0)}_{47}\\
&= \mathcal{A}^{(-1)}_{0,n_y+n_z}(\tilde{t})\mathcal{F}^{(1)}_{0,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\
&\quad+\mathcal{A}^{(0)}_{0,n_y+n_z}(\tilde{t})\mathcal{F}^{(0)}_{0,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\
&\quad+\mathcal{A}^{(1)}_{0,n_y+n_z}(\tilde{t})\mathcal{F}^{(-1)}_{0,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\
&\quad+\sum_{k_1=0}^{n_z}\sum_{k_2=0}^{n_y+k_1} {}_{n_z}C_{k_1\,n_y+k_1}C_{k_2}(-1)^{k_1+k_2}\left(\frac{s}{s-p_1^2}\right)^{n_y+k_1}\\
&\quad\times\left[\mathcal{A}^{(-1)}_{0,n_y+n_z}(\tilde{s})\left(\mathcal{F}^{(1)}_{0,0,k_2}\left(-\frac{\tilde{u}}{\bar{t}}\right)\right.\right.\\
&\quad-\left(\frac{p_1^2}{s}\right)^{k_2}\left(\mathcal{F}^{(1)}_{0,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right.\\
&\quad\left.\left.+\ln\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)\mathcal{F}^{(0)}_{0,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right)\right)\\
&\quad+\mathcal{A}^{(0)}_{0,n_y+n_z}(\tilde{s})\left(\mathcal{F}^{(0)}_{0,0,k_2}\left(-\frac{\tilde{u}}{\bar{t}}\right)\right.\\
&\quad-\left(\frac{p_1^2}{s}\right)^{k_2}\left(\mathcal{F}^{(0)}_{0,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right.\\
&\quad\left.\left.\left.+\ln\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)\mathcal{F}^{(-1)}_{0,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right)\right)\right].
\end{aligned}
\tag{B.66}
$$

B.4.3 $n_x \neq 0, n_y = 0, n_z = 0$

When $n_x \neq 0$, and $n_y = n_z = 0$, the final result can be written as

$$
\begin{aligned}
&J_4(s,t;p_1^2,0,0,0;n_x,0,0)\\
&= \frac{1}{(4\pi)^2 st}B(n_x+\varepsilon_{\rm IR},\varepsilon_{\rm IR})n_x!\Gamma(\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR})\\
&\quad\times\left[\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\left(\frac{-t}{s}\right)^{n_x}\frac{1}{\Gamma(n_x+\varepsilon_{\rm IR})}\mathcal{I}^{(1)}\right.\\
&\quad\left.+\left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\sum_{l=0}^{n_x}\frac{(-1)^l}{\Gamma(l+\varepsilon_{\rm IR})(n_x-l)!}\mathcal{I}^{(2)}_l\right].
\end{aligned}
\tag{B.67}
$$

The integrals $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ can be done easily:

$$
\begin{aligned}
\mathcal{I}^{(1)} &= B(n_x+\varepsilon_{\rm IR},1)\\
&\quad\times{}_2F_1\left(1+n_x,n_x+\varepsilon_{\rm IR},1+n_x+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\tilde{s}}\right),
\end{aligned}
\tag{B.68}
$$

$$
\begin{aligned}
\mathcal{I}^{(2)}_0 &= \frac{1}{\varepsilon_{\rm IR}}\left[{}_2F_1\left(1+l,l+\varepsilon_{\rm IR},1+l+\varepsilon_{\rm IR},-\frac{\tilde{u}}{\bar{t}}\right)\right.\\
&\quad\left.-\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)^{\varepsilon_{\rm IR}}{}_2F_1\left(1+l,l+\varepsilon_{\rm IR},1+l+\varepsilon_{\rm IR},-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right].
\end{aligned}
\tag{B.69}
$$

After an $\varepsilon_{\rm IR}$ expansion with the $\overline{\rm MS}$ scheme, it is obtained as

$$
\begin{aligned}
&J_4(s,t,p_1^2,0,0,0;n_x,0,0)\\
&\quad\to\frac{1}{(4\pi)^2 st}\left[\frac{\mathcal{C}^{(-2)}_{48}}{\varepsilon_{\rm IR}^2}+\frac{\mathcal{C}^{(-1)}_{48}}{\varepsilon_{\rm IR}}+\mathcal{C}^{(0)}_{48}+O(\varepsilon_{\rm IR})\right],
\end{aligned}
\tag{B.70}
$$

where

$$
\begin{aligned}
\mathcal{C}^{(-2)}_{48} &= n_x\left(-\frac{t}{s}\right)^{n_x}\mathcal{A}^{(-1)}_{0,n_x}(\tilde{t})\mathcal{F}^{(0)}_{n_x,0,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\
&\quad+\mathcal{A}^{(-1)}_{0,n_x}(\tilde{s})\left[\mathcal{F}^{(-1)}_{0,0,0}\left(-\frac{\tilde{u}}{\bar{t}}\right)-\mathcal{F}^{(-1)}_{0,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right.\\
&\quad\left.+\sum_{l=1}^{n_x}l(-1)^l{}_{n_x}C_l\left(\mathcal{F}^{(0)}_{l,0,0}\left(-\frac{\tilde{u}}{\bar{t}}\right)-\mathcal{F}^{(0)}_{l,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right)\right],
\end{aligned}
\tag{B.71}
$$

$$
\begin{aligned}
\mathcal{C}^{(-1)}_{48} &= n_x\left(-\frac{t}{s}\right)^{n_x}\\
&\quad\times\left[\mathcal{A}^{(-1)}_{0,n_x}(\tilde{t})\mathcal{F}^{(1)}_{n_x,0,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right)+\mathcal{A}^{(0)}_{0,n_x}(\tilde{t})\mathcal{F}^{(0)}_{n_x,0,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\right.\\
&\quad\left.-\mathcal{H}_{n_x-1}\mathcal{A}^{(-1)}_{0,n_x}(\tilde{t})\mathcal{F}^{(0)}_{n_x,0,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\right]\\
&\quad+\mathcal{A}^{(-1)}_{0,n_x}(\tilde{s})\left[\mathcal{F}^{(0)}_{0,0,0}\left(-\frac{\tilde{u}}{\bar{t}}\right)-\mathcal{F}^{(0)}_{0,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right.\\
&\quad\left.-\ln\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)\mathcal{F}^{(-1)}_{0,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right]\\
&\quad+\mathcal{A}^{(0)}_{0,n_x}(\tilde{s})\left[\mathcal{F}^{(-1)}_{0,0,0}\left(-\frac{\tilde{u}}{\bar{t}}\right)-\mathcal{F}^{(-1)}_{0,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right]\\
&\quad+\sum_{l=1}^{n_x}l(-1)^l{}_{n_x}C_l\left[\mathcal{A}^{(-1)}_{0,n_x}(\tilde{s})\right.\\
&\quad\times\left(\mathcal{F}^{(1)}_{l,0,0}\left(-\frac{\tilde{u}}{\bar{t}}\right)-\mathcal{F}^{(1)}_{l,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right.
\end{aligned}
\tag{B.72}
$$

$$
\begin{aligned}
&- \ln\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)\mathcal{F}^{(0)}_{l,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\Bigg)\\
&+\left(\mathcal{A}^{(0)}_{0,n_x}(\tilde{s})-\mathcal{A}^{(-1)}_{0,n_x}(\tilde{s})\mathcal{H}_{l-1}\right)\\
&\times\left(\mathcal{F}^{(0)}_{l,0,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)-\mathcal{F}^{(0)}_{l,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right)\Bigg], \qquad \text{(B.73)}\\
\mathcal{C}^{(0)}_{48} &= n_x\left(-\frac{t}{s}\right)^{n_x}\Bigg[\mathcal{A}^{(-1)}_{0,n_x}(\tilde{t})\mathcal{F}^{(2)}_{n_x,0,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\
&+\mathcal{A}^{(0)}_{0,n_x}(\tilde{t})\mathcal{F}^{(1)}_{n_x,0,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right)+\mathcal{A}^{(1)}_{0,n_x}(\tilde{t})\mathcal{F}^{(0)}_{n_x,0,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\
&-\mathcal{H}_{n_x-1}\left(\mathcal{A}^{(-1)}_{0,n_x}(\tilde{t})\mathcal{F}^{(1)}_{n_x,0,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\right.\\
&\left.+\mathcal{A}^{(0)}_{0,n_x}(\tilde{t})\mathcal{F}^{(0)}_{n_x,0,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\right)\\
&+\frac{1}{2}\left(\mathcal{H}_{n_x-1}+\frac{\pi^2}{6}-\psi^{(1)}_{n_x}\right)\mathcal{A}^{(-1)}_{0,n_x}(\tilde{s})\mathcal{F}^{(0)}_{n_x,0,0}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\Bigg]\\
&+\mathcal{A}^{(-1)}_{0,n_x}(\tilde{s})\left[\mathcal{F}^{(1)}_{0,0,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)-\mathcal{F}^{(1)}_{0,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right.\\
&\left.-\ln\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)\mathcal{F}^{(0)}_{0,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right]\\
&+\mathcal{A}^{(0)}_{0,n_x}(\tilde{s})\left[\mathcal{F}^{(0)}_{0,0,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)-\mathcal{F}^{(0)}_{0,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right.\\
&\left.-\ln\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)\mathcal{F}^{(-1)}_{0,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right]\\
&+\mathcal{A}^{(1)}_{0,n_x}(\tilde{s})\left[\mathcal{F}^{(-1)}_{0,0,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)-\mathcal{F}^{(-1)}_{0,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right]\\
&+\sum_{l=1}^{n_x} l(-1)^l{}_{n_x}C_l\Bigg[\mathcal{A}^{(-1)}_{0,n_x}(\tilde{s})\left(\mathcal{F}^{(2)}_{l,0,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)\right.\\
&\left.-\mathcal{F}^{(2)}_{l,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)-\ln\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)\mathcal{F}^{(1)}_{l,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right)\\
&+\left(\mathcal{A}^{(0)}_{0,n_x}(\tilde{s})-\mathcal{A}^{(-1)}_{0,n_x}(\tilde{s})\mathcal{H}_{l-1}\right)\\
&\times\left(\mathcal{F}^{(1)}_{l,0,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)-\mathcal{F}^{(1)}_{l,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right.\\
&\left.-\ln\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)\mathcal{F}^{(0)}_{l,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right)\\
&+\left(\mathcal{A}^{(1)}_{0,n_x}(\tilde{s})-\mathcal{A}^{(0)}_{0,n_x}(\tilde{s})\mathcal{H}_{l-1}\right.\\
&\left.+\mathcal{A}^{(-1)}_{0,n_x}(\tilde{s})\frac{1}{2}\left(\mathcal{H}_{l-1}+\frac{\pi^2}{6}-\psi^{(1)}_{l}\right)\right)\\
&\times\left(\mathcal{F}^{(0)}_{l,0,0}\left(-\frac{\tilde{u}}{\tilde{t}}\right)-\mathcal{F}^{(0)}_{l,0,0}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\right)\right)\Bigg]. \qquad \text{(B.74)}
\end{aligned}
$$

B.4.4 $n_x \neq 0, n_y + n_z \geq 1$

At last we treat the most general case with $n_x \neq 0$, and $n_y + n_z \geq 1$. The final result can be written as

$$
\begin{aligned}
J_4(s,t;p_1^2,0,0,0;n_x,n_y,n_z) &= \frac{1}{(4\pi)^2 st} \qquad \text{(B.75)}\\
&\times B(n_x+\varepsilon_{\rm IR}, n_y+n_z+\varepsilon_{\rm IR})n_x!\Gamma(\varepsilon_{\rm IR})\Gamma(1-\varepsilon_{\rm IR})\\
&\times\left[\left(\frac{-\tilde{t}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\left(\frac{-t}{s}\right)^{n_x}\frac{1}{\Gamma(n_x+\varepsilon_{\rm IR})}\mathcal{I}^{(1)}\right.\\
&\left.+\left(\frac{-\tilde{s}}{4\pi\mu_{\rm R}^2}\right)^{\varepsilon_{\rm IR}}\sum_{l=0}^{n_x}\frac{(-1)^l}{\Gamma(l+\varepsilon_{\rm IR})(n_x-l)!}\mathcal{I}^{(2)}_l\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}^{(1)} &= B(1+n_z, n_x+n_y+\varepsilon_{\rm IR})\\
&\quad\times{}_2F_1\Bigg(1+n_x, n_x+n_y+\varepsilon_{\rm IR},\\
&\qquad 1+n_x+n_y+n_z+\varepsilon_{\rm IR}, -\frac{\tilde{u}}{\tilde{s}}\Bigg), \qquad \text{(B.76)}\\
\mathcal{I}^{(2)}_l &= \sum_{k_1=0}^{n_z}\sum_{k_2=0}^{n_y+k_1}{}_{n_z}C_{k_1\,n_y+k_1}C_{k_2}(-1)^{k_1+k_2}\\
&\quad\times\left(\frac{s}{s-p_1^2}\right)^{n_y+k_1}\frac{1}{l+k_2+\varepsilon_{\rm IR}}\left(1+\frac{u}{t}\right)^l\\
&\quad\times\Bigg[{}_2F_1\left(1+l, l+k_2+\varepsilon_{\rm IR}, 1+l+k_2+\varepsilon_{\rm IR}, -\frac{\tilde{u}}{\tilde{t}}\right)\\
&\quad-\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)^{l+k_2+\varepsilon_{\rm IR}}\\
&\quad\times{}_2F_1\Bigg(1+l, l+k_2+\varepsilon_{\rm IR},\\
&\qquad 1+l+k_2+\varepsilon_{\rm IR}, -\frac{\tilde{u}\tilde{p_1}^2}{\tilde{t}\tilde{s}}\Bigg)\Bigg], \qquad \text{(B.77)}
\end{aligned}
$$

In this case, the result might be IR finite. Then, after an $\varepsilon_{\rm IR}$ expansion with the $\overline{\rm MS}$ scheme, it becomes

$$
J_4(s,t,p_1^2,0,0,0;n_x,n_y,n_z) \to \frac{1}{(4\pi)^2 st}\left[\mathcal{C}^{(0)}_{49}+O(\varepsilon_{\rm IR})\right], \qquad \text{(B.78)}
$$

where

$$
\begin{aligned}
\mathcal{C}^{(0)}_{49} &= n_x\left(-\frac{t}{s}\right)^{n_x}\\
&\quad\times\Bigg[\mathcal{A}^{(0)}_{n_x,n_y+n_z}(\tilde{t})\mathcal{F}^{(1)}_{n_x,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\
&\quad+\mathcal{A}^{(1)}_{n_x,n_y+n_z}(\tilde{t})\mathcal{F}^{(0)}_{n_x,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\\
&\quad-\mathcal{H}_{n_x-1}\mathcal{A}^{(0)}_{n_x,n_y+n_z}(\tilde{t})\mathcal{F}^{(0)}_{n_x,n_z,n_y}\left(-\frac{\tilde{u}}{\tilde{s}}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k_1=0}^{n_z}\sum_{k_2=0}^{n_y+k_1} {}_{n_z}C_{k_1\,n_y+k_1}C_{k_2}(-1)^{k_1+k_2}\left(\frac{s}{s-p_1^2}\right)^{n_y+k_1}\\
&\times\Bigg[\mathcal{A}^{(0)}_{n_x,n_y+n_z}(\tilde{s})\\
&\times\left(\mathcal{F}^{(0)}_{0,0,k_2}\left(-\frac{\tilde{u}}{\bar{t}}\right)-\left(\frac{p_1^2}{s}\right)^{k_2}\left(\mathcal{F}^{(0)}_{0,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right.\right.\\
&\left.\left.+\ln\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)\mathcal{F}^{(-1)}_{0,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right)\right)\\
&+\mathcal{A}^{(1)}_{n_x,n_y+n_z}(\tilde{s})\\
&\times\left(\mathcal{F}^{(-1)}_{0,0,k_2}\left(-\frac{\tilde{u}}{\bar{t}}\right)-\left(\frac{p_1^2}{s}\right)^{k_2}\mathcal{F}^{(-1)}_{0,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right)\Bigg]\\
&+\sum_{l=1}^{n_x} l(-1)^l{}_{n_x}C_l\left(1+\frac{u}{t}\right)^l\\
&\times\sum_{k_1=0}^{n_z}\sum_{k_2=0}^{n_y+k_1} {}_{n_z}C_{k_1\,n_y+k_1}C_{k_2}(-1)^{k_1+k_2}\left(\frac{s}{s-p_1^2}\right)^{n_y+k_1}\\
&\times\Bigg[\mathcal{A}^{(0)}_{n_x,n_y+n_z}(\tilde{s})\\
&\times\left(\mathcal{F}^{(1)}_{l,0,k_2}\left(-\frac{\tilde{u}}{\bar{t}}\right)-\left(\frac{p_1^2}{s}\right)^{l+k_2}\left(\mathcal{F}^{(1)}_{l,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right.\right.\\
&\left.\left.+\ln\left(\frac{\tilde{p_1}^2}{\tilde{s}}\right)\mathcal{F}^{(0)}_{l,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right)\right)\\
&\times\left(\mathcal{A}^{(1)}_{n_x,n_y+n_z}(\tilde{s})-\mathcal{A}^{(0)}_{n_x,n_y+n_z}(\tilde{s})\mathcal{H}_{l-1}\right) \qquad \text{(B.79)}\\
&\times\left(\mathcal{F}^{(0)}_{l,0,k_2}\left(-\frac{\tilde{u}}{\bar{t}}\right)-\left(\frac{p_1^2}{s}\right)^{l+k_2}\mathcal{F}^{(0)}_{l,0,k_2}\left(-\frac{\tilde{u}\tilde{p_1}^2}{\bar{t}\tilde{s}}\right)\right)\Bigg].
\end{aligned}
$$

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